

Solutions Manual
Applied Mathematics, 3rd Edition

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Preface

This manual contains hints or full solutions to many of the problems in Chapters 1, 2, and 3 of the text: J. David Logan, 2006. *Applied Mathematics*, 3rd ed., Wiley–Interscience, New York.

I would like to thank Glenn Ledder, my colleague at UNL, who has taught the course many times and who has been the source of many examples, exercises, and suggestions.

Comments and corrections will be greatly appreciated.

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Chapter 1

Scaling, Dimensional Analysis (Secs. 1 & 2)

1.1 Dimensional Analysis

Exercises, page 7

1. The period cannot depend only on the length and mass; there is no way that length and mass can be combined to yield a time dimension. If we assume there is a physical law $f(P, L, g) = 0$, then $P = F(L, g)$. The right side must be time dimensions, and the only way that we can get time dimensions with g and L is to take $\sqrt{L/g}$. Thus, $P = C\sqrt{L/g}$ for some constant C .
2. If $f(D, e) = 0$ then we can solve and get $e = F(D)$. Now, e is energy per mass, or length-squared per time-squared. So the right hand side of the equation must be proportional to D^2 . Then $e = cD^2$ for some constant c .
3. Using nonlinear regression, write

$$r = bt^{2/5}, \quad b = (E/\rho)^{1/5},$$

The sum of the squares of the errors is

$$S = \sum_{i=1}^8 (bt_i^{2/5} - r_i)^2.$$

Take the derivative with respect to b and set it equal to zero to get

$$b = \frac{\sum r_i t_i^{0.4}}{\sum t_i^{0.8}} = 569.5695.$$

Then the energy in kilotons is

$$E = \frac{1}{4.186 \times 10^{12}} \rho b^5 = 17.89.$$

Another method is to average. We have

$$E = \frac{\rho r^5}{t^2}.$$

Substitute each data point to get

$$E_i = \frac{1.25r_i^5}{t_i^2}, \quad i = 1, 2, \dots, 8.$$

Now average the E_i and divide by 4.186×10^{12} to get $E = 18.368$.

4. The variables are t, r, ρ, e, P . We already know one dimensionless quantity $\pi_1 = \rho r^5 / et^2$. Try to find another that uses P , which is a pressure, or force per unit area, that is, mass per length per time-squared. By inspection,

$$\pi_2 = \frac{P}{\rho g r}$$

is another dimensionless quantity. Thus we have

$$f(\rho r^5 / et^2, \frac{P}{\rho g r}) = 0.$$

Now we cannot isolate the r and t variables in one dimensionless expression. If we solve for the first dimensionless quantity we get

$$\rho r^5 / et^2 = F\left(\frac{P}{\rho g r}\right).$$

Then

$$r = \left(\frac{et^2}{\rho}\right)^{1/5} F\left(\frac{P}{\rho g r}\right).$$

Because the second dimensionless variable contains r in some unknown manner, we cannot conclude that r varies like $t^{2/5}$. However, if one can argue that the ambient pressure is small and can be neglected, then we can set $P = 0$ and obtain the result

$$r = \left(\frac{et^2}{\rho}\right)^{1/5} F(0),$$

which does imply that r varies like $t^{2/5}$.

5. If $x = \frac{1}{2}gt^2$, then $\pi = x/gt^2$ is dimensionless and the physical law is $\pi = \frac{1}{2}$. If we include mass, then m must be some function of t, x, g , which is impossible.
6. If $x = -\frac{1}{2}gt^2 + vt$, then, by inspection, $y = x/gt^2$ and $s = v/gt$ are dimensionless. Dividing the equation by gt^2 gives the dimensionless form $y = -1/2 + s$.

Exercises, page 17

1. Assume that $f(v, \Lambda, g) = 0$. If π is dimensionless

$$\begin{aligned} [\pi] &= [v^{\alpha_1} \Lambda^{\alpha_2} g^{\alpha_3}], \\ &= (LT^{-1})^{\alpha_1} L^{\alpha_2} (LT^{-2})^{\alpha_3}. \end{aligned}$$

Thus we have the homogeneous system

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad -\alpha_1 - 2\alpha_3 = 0.$$

The rank of the coefficient matrix is one, so there is one dimensionless variable. Notice that $(-2, 1, 1)$ is a solution to the system, and thus

$$\pi = \Lambda g / v^2.$$

By the Pi theorem, $F(\pi) = 0$ or $\Lambda g / v^2 = \text{Const.}$

2. Two dimensionless variables are

$$\frac{\rho V}{m}, \quad \frac{S}{V^{2/3}}.$$

Therefore

$$\frac{\rho V}{m} = f\left(\frac{S}{V^{2/3}}\right).$$

3. Pick length, time and mass as fundamental and write

$$\bar{x} = \lambda_1 x, \quad \bar{t} = \lambda_2 t, \quad \bar{m} = \lambda_3 m.$$

Then write $\bar{v} = \lambda_1 \lambda_2^{-1} v$, and so on for the other variables. Show that

$$\bar{v} - \frac{2}{9} \bar{r}^2 \bar{\rho} \bar{g} \bar{\mu}^{-1} (1 - \bar{\rho}_l / \bar{\rho}) = \lambda_1 \lambda_2^{-1} (v - \frac{2}{9} r^2 \rho g \mu^{-1} (1 - \rho_l / \rho))$$

So, by definition, the law is unit free.

4. Select M , L , and T (mass, length, and time) as fundamental dimensions.
 5. There is only one dimensionless variable among E , P , and A , namely $PA^{3/2}/E$. Thus,

$$PA^{3/2}/E = \text{const.}$$

6. The two dimensionless variables are

$$\frac{at}{\rho L}, \quad \frac{bt}{\rho}.$$

7. Dimensionless quantities are

$$\frac{\rho}{\rho_e}, \quad \frac{E}{v^2}.$$

Thus, by the pi theorem,

$$v = \sqrt{E} f\left(\frac{\rho}{\rho_e}\right).$$

8. Select M , L , and T (mass, length, and time) as fundamental dimensions.

9. Assume there is a physical law $f(T, V, C, Y, r) = 0$. We have

$$\pi = T^{\alpha_1} V^{\alpha_2} C^{\alpha_3} Y^{\alpha_4} r^{\alpha_5}$$

and so

$$1 = T^{\alpha_1} (L^3)^{\alpha_2} (ML^{-3})^{\alpha_3} (MT^{-1})^{\alpha_4} (MT^{-1}V^{-3})^{\alpha_5}.$$

Setting the powers of T , L , and M equal to zero and solving gives

$$\alpha_1 = \alpha_4 + \alpha_5, \quad \alpha_2 = -\alpha_4, \quad \alpha_3 = -\alpha_4 - \alpha_5.$$

This leads to two dimensionless quantities

$$\frac{TY}{VC}, \quad \frac{Tr}{C}.$$

10. Select M , L , and T (mass, length, and time) as fundamental dimensions.

11. Select M , L , and T (mass, length, and time) as fundamental dimensions.

12. Pick length L , time T , and mass M as fundamental dimensions. Then the dimension matrix has rank three and there are $5 - 3 = 2$ dimensionless variables; they are given by $\pi_1 = \gamma$ and $\pi_2 = R\omega\sqrt{\rho l}/\sqrt{P}$. Thus $f(\pi_1, \pi_2) = 0$ implies

$$\omega = R^{-1}\sqrt{P/\rho l}G(\gamma)$$

for some function G .

13. The dimensions are

$$[E] = \frac{\text{energy}}{\text{mass}}, \quad [T] = \text{temp}, \quad [k] = \frac{\text{energy}}{\text{mass temp}}.$$

It is clear there is only one dimensionless variable, $\pi = E/kT$. Thus $E/kT = \text{const.}$

14. We have dimensions

$$[F] = M L T^{-1}, [V] = L T^{-1}, [C] = L^3 T^{-1}, [K] = M L^{-1} T^{-2}.$$

Assume a physical law $f(F, V, C, K) = 0$. If π is dimensionless, then

$$\pi = F^{\alpha_1} V^{\alpha_2} C^{\alpha_3} K^{\alpha_4}.$$

This gives

$$1 = (M L T^{-1})^{\alpha_1} (L T^{-1})^{\alpha_2} (L^3 T^{-1})^{\alpha_3} (M L^{-1} T^{-2})^{\alpha_4}.$$

This leads to the system of equations

$$\begin{aligned} \alpha_1 + \alpha_2 + 3\alpha_3 - \alpha_4 &= 0, \\ -2\alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 &= 0, \\ \alpha_1 + \alpha_4 &= 0. \end{aligned}$$

This system has rank 3 and so there is one solution, $(-1, -1, 1, 1)$, which gives the dimensionless variable $CK/FV = \text{const}$.

15. We have dimensions

$$[w] = L, [C_0] = [C - 1] = M L^{-3}, [d] = L^2 T^{-1}, [\phi] = M L^{-2} T^{-1}.$$

Assume a physical law $f(w, C_0, C_1, d, \phi) = 0$. If π is dimensionless, then

$$\pi = w^{\alpha_1} C_0^{\alpha_2} C_1^{\alpha_3} d^{\alpha_4} \phi^{\alpha_5}.$$

This gives

$$1 = L^{\alpha_1} (M L^{-3})^{\alpha_2} (M L^{-3})^{\alpha_3} (L^2 T^{-1})^{\alpha_4} (M L^{-2} T^{-1})^{\alpha_5}.$$

This leads to the system of equations

$$\begin{aligned} \alpha_1 - 3\alpha_2 - 3\alpha_3 + 2\alpha_4 - 2\alpha_5 &= 0, \\ \alpha_2 + \alpha_3 + \alpha_5 &= 0, \\ \alpha_4 + \alpha_5 &= 0. \end{aligned}$$

The system has rank 3 and so there are two independent solutions $(0, -1, 1, 0, 0)$ and $(1, -1, 0, -1, 1)$. This gives dimensionless variables

$$\frac{C_0}{C_1}, \quad \frac{w\phi}{dC_0}.$$

Therefore

$$\frac{w\phi}{dC_0} = G\left(\frac{C_0}{C_1}\right),$$

which gives the form of the flux ϕ ,

$$\phi = \frac{dC_0}{w} G\left(\frac{C_0}{C_1}\right).$$

1.2 Scaling

Exercises, page 30

- In (a) we have $u = A \sin \omega t$ and so $u' = \omega A \cos \omega t$. Then $M = A$ and $\max |u'| = \omega A$. Then we have $t_c = 1/\omega$. In (b) we have $u = Ae^{-\lambda t}$ and $u' = -\lambda Ae^{-\lambda t}$. Then $t_c = \max |u| / \max |u'| = 1/\lambda$. In part (c) we have $u = Ate^{-\lambda t}$ and $u' = (1 - \lambda t)Ae^{-\lambda t}$. The maximum of u occurs at $t = 1/\lambda$ and is $M = A/\lambda e$. To find the maximum of u' we calculate the second derivative to get $u'' = A\lambda(\lambda t - 2)e^{-\lambda t}$. So the maximum derivative occurs at $t = 2/\lambda$ or at an endpoint. It is easily checked that the maximum derivative occurs at $t = 0$ and has value $\max |u'| = A$ on the given interval. Therefore $t_c = (A/\lambda e)/A = 1/\lambda e$.
- Here $u = 1 + \exp(-t/\epsilon)$ and $u' = -\exp(-t/\epsilon)/\epsilon$. Then $t_c = \max |u| / \max |u'| = 2/\epsilon^{-1} = 2\epsilon$. The time scale is very small, indicating rapid change in a small interval. But a graph shows that that this rapid decrease occurs only in a small interval near $t = 0$; in most of the interval the changes occur slowly. Thus two time scales are suggested, one near the origin and one out in the interval where t is order one.

- We have

$$m' = ax^2 - bx^3, \quad m = \rho x^3.$$

Thus

$$(\rho x^3)' = 3x^2 \rho x' = ax^2 - bx^3,$$

giving

$$x' = \frac{a}{3\rho} - \frac{b}{3\rho}x.$$

We have a given in mass per time per length-squared and b in mass per time per volume. Scaling time by ρ/b and length by a/b leads to the dimensionless model

$$y' = \frac{1}{3} - \frac{1}{3}y.$$

If $x(0) = 0$ then $y(0) = 0$ and the solution to the dimensionless model is

$$y(\tau) = 1 - e^{\tau/3}.$$

Yes, this is a reasonable model. The organism grows exponentially toward a limiting value. This is, in fact, observed with most organisms.

- The constants in the problem, V , k , and a have dimensions

$$[V] = \frac{L}{T}, \quad [k] = \frac{M}{T^2}, \quad [a] = \frac{M}{TL}.$$

One time scale is $\sqrt{m/k}$ which is based on damping. Another is $\sqrt{m/aV}$, which is based on the restoring force. To rescale, let T and L be scales to

be chosen and let $y = x/L$ and $\tau = t/T$. Then the model becomes

$$y'' = -\frac{aTL}{m}y|y'| - \frac{T^2k}{m}y, \quad y(0) = 0, \quad y'(0) = \frac{VT}{L}.$$

We want the restoring force to be small and have the small coefficient. Therefore take $T = \sqrt{m/aV}$. Then we get

$$y'' = -\frac{aL\sqrt{m/aV}}{m}y|y'| - \frac{k}{aV}y, \quad y(0) = 0, \quad y'(0) = \frac{V\sqrt{m/aV}}{L}$$

Now choose the length scale L so that the coefficient of $y|y'|$ is one. The differential equation then becomes

$$y'' = -y|y'| - \epsilon y, \quad y(0) = 0, \quad y'(0) = 1, \quad \epsilon \equiv \frac{k}{aV}.$$

So, the small coefficient is in front of the small damping term.

5. The dimensions of the constants are

$$[I] = \frac{ML}{T}, \quad [a] = \frac{M}{T}, \quad [k] = \frac{M}{T^2L}.$$

Letting $u = x/(I/a)$ and $\tau = t/T$, where T is yet to be determined, we get

$$\frac{m}{aT}u'' = -u' - \frac{kTI^2}{a^3}u, \quad u(0) = 0, \quad \frac{m}{aT}u'(0) = 1.$$

If the mass is small, we want to choose T so that the coefficient of the u'' term is small. So, select T that makes the restoring force term have coefficient 1. Thus, take

$$T = \frac{a^3}{kI^2}.$$

The model then becomes

$$\epsilon u'' = -u' - u, \quad \epsilon = \frac{mkI^2}{a^4}.$$

6. (a) The constant a must be budworms-squared because it is added to such a term in the denominator. The entire predation term must be budworms per time, and so b must have dimensions budworms per time. (b) The parameter a defines the place where the predation term makes a significant rise. Thus it indicates the threshold where the number of budworms is plentiful so that predation kicks in; there are enough budworms to make the birds interested. (c) The dimensionless equation is

$$\frac{dN}{d\tau} = sN(1 - N/q) - \frac{N^2}{1 + N^2}.$$

(d) To find equilibrium solutions we set

$$sN(1 - N/q) - \frac{N^2}{1 + N^2} = 0.$$

At this point we can use a calculator or a computer algebra program like Maple or Mathematica to solve the equation for N . Observe that we obtain a fourth degree polynomial equation when we simplify this algebraic equation:

$$sN(1 - N/q)(1 + N^2) - N^2 = 0.$$

When $s = 12$ and $q = 0.25$ the equilibrium populations are $N = 0, /, 0.261$.

When $s = 0.4$ and $q = 35$ the equilibrium populations are $N = 0, 0.489, 2.218, 32.29$.

7. Introduce the following dimensionless variables:

$$\bar{m} = m/M, \quad \bar{x} = x/R, \quad \bar{t} = t/T, \quad \bar{v} = v/V,$$

where T and V are to be determined. In dimensionless variables the equations now take the form

$$\begin{aligned} \bar{m}' &= -\frac{\alpha T}{M}, \\ \bar{x}' &= -\frac{VT}{R}\bar{v}, \\ \bar{v}' &= \frac{\alpha\beta T}{MV}\frac{1}{\bar{m}} - \frac{Tg}{V(1-\bar{x})^2}. \end{aligned}$$

To ensure that the terms in the velocity and acceleration equations are the same order, with the gravitational term small, pick

$$\frac{VT}{R} = \frac{\alpha\beta T}{MV},$$

which gives

$$V = \sqrt{\alpha\beta R/M}$$

as the velocity scale.

8. The differential equation is

$$c' = -\frac{q}{V}(c_i - c) - kc^2, \quad c(0) = c_0.$$

Here k is a volume per mass per time. Choosing dimensionless quantities via

$$C = c/c_i, \quad \tau = t/(V/q),$$

the model equation becomes

$$\frac{dC}{d\tau} = -(1 - C) - bC^2, \quad C(0) = \gamma,$$

where $\gamma = c_0/c_i$ and $b = kVc_i/q$. Solve this initial value problem using separation of variables.

9. The quantity q is degrees per time, k is time^{-1} , and θ is degrees (one can only exponentiate a pure number). Introducing dimensionless variables

$$\bar{T} = T/T_f, \quad \tau = t/(T_f/q),$$

we obtain the dimensionless model

$$\frac{d\bar{T}}{d\tau} = e^{-E/\bar{T}} - \beta(1 - \bar{T}), \quad T(0) = \alpha,$$

where $\alpha = T_0/T_f$, $E = \theta/T_f$, and $\beta = kT_f/q$.

10. Let h be the height measured above the ground. Then Newton's second law gives

$$mh'' = -mg - a(h')^2, \quad h(0) = 0, \quad h'(0) = V.$$

Choose new dimensionless time and distance variables according to

$$\tau = t/(V/g), \quad y = h/(V^2/g).$$

Then the dimensionless model is

$$y'' = -1 - \alpha(y')^2, \quad y(0) = 0, \quad y'(0) = 1,$$

where prime is a τ derivative and $\alpha = aV^2/mg$.

11. The model is

$$mx'' = -\frac{k}{x^2}e^{-t/a}, \quad x(0) = L, \quad x'(0) = 0.$$

We have $[a] = T$ and $[k] = \frac{ML^3}{T^2}$. Two time scales are

$$a, \quad \sqrt{\frac{mL^3}{k}}.$$

12. The model is

$$mx'' = -kxe^{-t/a}, \quad x(0) = L, \quad x'(0) = V.$$

Let $\tau = t/a$ and $y = x/L$ be dimensionless variables. Then

$$y'' = -\alpha ye^{-\tau}, \quad y(0) = 1, \quad y'(0) = \beta,$$

where $\alpha = -ka^2/m$ and $\beta = Va/L$.

13. The dynamics is given by

$$x' = rx(1 - x/K) \quad \text{if } t < t_f,$$

and

$$x' = rx(1 - x/K) - qb(t)x \quad \text{if } t > t_f.$$

Nondimensionalizing,

$$y' = y(1 - y) \quad \text{if } \tau < \tau_f,$$

and

$$y' = y(1 - y) - \beta y \quad \text{if } \tau > \tau_f.$$

Here, $y = x/K$ and $\tau = rt$. Also, $\beta = qB/r$ where $b(t) = B$.

Setting $y' = 0$ for $\tau > \tau_f$ gives the constant population $y = 1 - \beta$. If we solve for $y = y(\tau)$ for $\tau < \tau_f$, then we can set $y(\tau_f) = 1 - \beta$ to obtain τ_f as a function of β .

14. The mass times acceleration is the force, or

$$ms'' = -mg \sin \theta,$$

where the force is the tangential component of the force mg along the arc of the path. But $s = L\theta$, and so

$$mL\theta'' = -mg \sin \theta.$$

Then

$$\theta'' + \frac{g}{L} \sin \theta = 0, \quad \theta(0) = \theta_0, \quad \theta'(0) = 0.$$

We can scale theta by θ_0 and time by $\sqrt{L/g}$. Then the model becomes

$$\psi'' + \frac{1}{\theta_0} \sin(\theta_0 \psi) = 0, \quad \psi(0) = 1, \quad \psi'(0) = 0.$$

15. Three time scales are

$$\sqrt{L/g}, \quad \omega_0^{-1}, \quad \frac{1}{k}.$$

These time scales involve the effect of gravity (undamped oscillations), the angular frequency caused by the initial angular velocity, and the time scale of the damping.

Chapter 2

Perturbation Methods

2.1 Regular Perturbation

Exercises, page 100

1. Since the mass times the acceleration equals the force, we have $my'' = -ky - a(y')^2$. The initial conditions are $y(0) = A, y'(0) = 0$. Here, a is assumed to be small. The scale for y is clearly the amplitude A . For the time scale choose $\sqrt{m/k}$, which is the time scale when *no* damping is present. Letting $\bar{y} = y/A, \tau = t/\sqrt{m/k}$ be new dimensionless variables, the model becomes

$$\bar{y}'' + \epsilon(\bar{y}')^2 + \bar{y} = 0,$$

where $\epsilon \equiv aA/m$. The initial data is $\bar{y}(0) = 1, \bar{y}'(0) = 0$. Observe that the small parameter is on the resistive force term, which is correct.

2. The problem is

$$u'' - u = \epsilon tu, \quad u(0) = 1, \quad u'(0) = -1.$$

A two-term perturbation expansion is given by

$$y(t) = e^{-t} + \frac{1}{8}\epsilon(e^t - e^{-t}(1 + 2x + 2x^2)).$$

A six-term Taylor expansion is

$$y(t) = 1 - t + \frac{1}{2}t^2 + \frac{1-\epsilon}{6}x^3 + \frac{1-2\epsilon}{24}x^4 - \frac{1-4\epsilon}{120}x^5.$$

Plots show the superior performance of the two-term perturbation approximation.

3. We have $e^{-t} = o(\frac{1}{t^2})$ as $t \rightarrow \infty$ because (using L'Hopital's rule)

$$\lim_{t \rightarrow \infty} \frac{e^{-t}}{\frac{1}{t^2}} = \lim_{t \rightarrow \infty} t^2 e^{-t} = 0.$$

4. Using the binomial theorem,

$$(1 + \varepsilon y)^{-3/2} = 1 - \frac{3}{2}\varepsilon y + \frac{(-3/2)(-5/2)}{2!}\varepsilon^2 y^2 + \dots$$

Now substitute $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$ and expand.

5. (a) We have

$$\frac{t^2 \tanh t}{t^2} = \tanh t < 1$$

for large t . Thus $t^2 \tanh t = 0(t^2)$ as $t \rightarrow \infty$.

(b) We have

$$\lim_{t \rightarrow \infty} \frac{e^{-t}}{1} = 0,$$

which proves the order relation.

(c) The order relation follows from the inequality

$$\frac{\sqrt{\varepsilon(1-\varepsilon)}}{\sqrt{\varepsilon}} = \sqrt{1-\varepsilon} \leq 1$$

for small, positive ε .

(d) The idea is to expand $\cos \varepsilon$ in a power series to get

$$\begin{aligned} & \frac{\sqrt{\varepsilon}}{1 - (1 - \varepsilon^2/2 + \varepsilon^4/4! - \dots)} \\ &= \frac{\sqrt{\varepsilon}}{\varepsilon^2/2 - \varepsilon^4/4! + \dots} \\ &= \frac{1}{2}\varepsilon^{-3/2} \frac{1}{1 - \varepsilon^2/12 + \dots}. \end{aligned}$$

But, using the geometric series,

$$\frac{1}{1 - \varepsilon^2/12 + \dots} = 1 + 0(\varepsilon^2),$$

which gives the result.

(e) We have

$$\frac{t}{t^2} = \frac{1}{t} \leq 1$$

for large t . So the ratio is bounded, proving the assertion.

(f) By Taylor's expansion,

$$e^\varepsilon - 1 = 1 + \varepsilon + 0(\varepsilon^2) - 1 = 0(\varepsilon).$$

(g) Expand the integrand in a Taylor series and integrate term-by-term to get

$$\begin{aligned}\int_0^\epsilon e^{-x^2} dx &= \int_0^\epsilon \left(1 - x^2 + \frac{x^2}{2!} + \dots\right) dx \\ &= \epsilon - \frac{1}{3}\epsilon^3 + \dots \\ &= 0.(\epsilon)\end{aligned}$$

An alternate method is to notice that $e^{-x^2} \leq 1$, which gives $\int_0^\epsilon e^{-x^2} dx \leq \epsilon$. Thus

$$\frac{\int_0^\epsilon e^{-x^2} dx}{\epsilon} \leq 1$$

(h) Observe that

$$\lim_{\epsilon \rightarrow 0} e^{\tan \epsilon} = 1.$$

Because the limit exists, the function must be bounded in a neighborhood of $\epsilon = 0$, which implies the result.

(i) Notice that

$$\frac{e^{-\epsilon}}{\epsilon^{-p}} = e^{-\epsilon} \epsilon^p \rightarrow 0$$

as $\epsilon \rightarrow \infty$ (exponentials decay faster than power functions grow). One can use L'Hospital's rule to show this.

(j) Notice that

$$\frac{\ln \epsilon}{\epsilon^{-p}} = \epsilon^p \ln \epsilon \rightarrow 0$$

as $\epsilon \rightarrow 0$ since power functions go to zero faster than logarithms grow near zero. Again, use L'Hospital's rule to verify this fact.

6. Substitute $x = x_0 + \epsilon x_1 + \dots$ into the nonlinear equation to get

$$h(\epsilon) \equiv \phi(x_0 + \epsilon x_1 + \dots, \epsilon) = 0$$

By Taylor's expansion

$$h(\epsilon) = h(0) + h'(0)\epsilon + \frac{1}{2}h''(0)\epsilon^2 + \dots$$

Using the chain rule we can compute these derivatives of h at $\epsilon = 0$ and thus expand the equation in powers of ϵ . We get

$$\phi(x_0) = 0, \quad x_1 = -\frac{\phi_\epsilon(x_0, 0)}{\phi_x(x_0, 0)},$$

and so on. To obtain x_1 we clearly require $\phi_x(x_0, 0) \neq 0$.

7. $(x + 1)^3 = \varepsilon x$. Set $x = x_0 + x_1\varepsilon + \dots$ and expand.

8. (a) Let $\tau = \omega t$, with $\omega = 1 + \omega_1\varepsilon + \dots$. Then the problem becomes

$$\omega^2 y'' + y = \varepsilon y \omega^2 (y')^2, \quad y(0) = 1, \quad \omega y'(0) = 0.$$

Here, prime denotes a τ derivative. Now assume a regular perturbation expansion. The leading order problem is

$$y_0'' + y_0 = 0, \quad y_0(0) = 1, \quad y_0'(0) = 0,$$

which has solution

$$y_0(\tau) = \cos \tau.$$

The next order problem is

$$y_1'' + y_1 = -2\omega_1 y_0'' + y_0 (y_0')^2, \quad y_1(0) = 0, \quad y_1'(0) = 0.$$

The equation simplifies to

$$y_1'' + y_1 = \left(\frac{1}{4} + 2\omega_1\right) \cos \tau - \frac{1}{4} \cos 3\tau.$$

To eliminate the secular term take $\omega_1 = -1/8$. Then, to leading order,

$$y_0(t) = \cos \left(\left(1 - \frac{1}{8}\varepsilon\right) t \right).$$

(b) Let $\tau = \omega t$, with $\omega = 3 + \omega_1\varepsilon + \dots$. Then the problem becomes

$$\omega^2 y'' + 9y = 3\varepsilon y^3, \quad y(0) = 0, \quad \omega y'(0) = 1.$$

Here, prime denotes a τ derivative. Now assume a regular perturbation expansion. The leading order problem is

$$y_0'' + y_0 = 0, \quad y_0(0) = 0, \quad y_0'(0) = \frac{1}{3},$$

which has solution

$$y_0(\tau) = \frac{1}{3} \sin \tau.$$

The next order problem is

$$9y_1'' + 9y_1 = 3y_0^2 - 6\omega_1 y_0'', \quad y_1(0) = 0, \quad y_1'(0) = -\frac{1}{9}.$$

The equation simplifies to

$$y_1'' + y_1 = \frac{1}{9} \left(\frac{1}{12} + 2\omega_1 \right) \sin \tau - \frac{1}{9 \cdot 36} \sin 3\tau.$$

To eliminate the secular term take $\omega_1 = -1/24$. Then, to leading order,

$$y_0(t) = \frac{1}{3} \sin \left(\left(3 - \frac{1}{24}\varepsilon\right) t \right).$$

9. The equation with ϵ can be handled with a perturbation series in ϵ . After determining the coefficients, one can substitute $\epsilon = 0.001$.
10. If we ignore $0.01x$ in the first equation then $y = 0.1$. Then, from the second equation $x = 0.9$. Checking these values by substituting back into the first equation gives $0.01(0.9) + 0.1 = 0.109$, so the approximation appears to be good. But the exact solution is $x = 190$, $y = 1$. So the approximation is in fact terrible. What went wrong? Since $x = -90$ the first term in the first equation is $0.01x = -0.9$, which is not small compared to the two other terms in the first equation. Thus the first term cannot be neglected.
11. Letting $h = h_0 + h_1\epsilon + \dots$ gives

$$h_0'' + \epsilon h_1'' + \epsilon^2 h_2'' + \dots = -1 + 2h_0\epsilon - (3h_0 - 2h_1)\epsilon^2 + \dots$$

Here we used the binomial theorem to expand the right hand side. We also have the initial conditions

$$h_0(0) = 0, \quad h_0'(0) = 1; \quad h_1(0) = h_1'(0) = 0, \dots$$

The equations are

$$h_0'' = -1, \quad h_1'' = 2h_0, \quad h_2'' = -(3h_0 - 2h_1), \dots$$

Now we can solve consecutively and get h_0, h_1, h_2, \dots . Once the expansion is obtained, solve $h'(t) = 0$ to get t_m .

12. The initial value problem is

$$my'' = -ay' - kye^{-rt}, \quad y(0) = y_0, \quad y'(0) = 0.$$

The quantity m is mass, y_0 is length, r is time^{-1} , a is mass per time, and k is mass per time-squared. To nondimensionalize, take $u = y/y_0$ and $\tau = t/(m/a)$. Then

$$u'' = -u' - \epsilon ue^{-\alpha\tau}, \quad u(0) = 1, \quad u'(0) = 0,$$

where

$$\epsilon = -\frac{km}{a^2}, \quad \alpha = \frac{mr}{a}.$$

To leading order we have $u_0'' = -u_0'$ which gives $u_0(t) = A + Be^{-\tau}$. From the initial conditions $A = 1$ and $B = 0$, giving

$$u_0(t) = 1.$$

At the next order we have $u_1'' = -u_1' - e^{-\alpha\tau}$ with zero initial conditions. The general solution is

$$u_1(\tau) = A + Be^{-\tau} + \frac{1}{\alpha - \alpha^2} e^{-\alpha\tau}.$$

Use the initial conditions to determine A and B . (Here we are assuming $\alpha \neq 1$.) Continue in this manner.

13. Consider the equation

$$y'' = \varepsilon y \cos \frac{\pi x}{2}.$$

Assuming a perturbation expansion, we get

$$y_0'' = 0, \quad y_1'' = y_0 \cos \frac{\pi x}{2}, \dots$$

Easily, $y_0(t) = x$, and so on.

14. We have

$$y_0' + \varepsilon y_1' + \dots = e^{-\varepsilon ps / (y_0 + \varepsilon y_1 + \dots)}$$

To leading order we have $y_0' = 1$ which gives, using the initial condition, $y_0(t) = x + 1$. To get higher order terms, use the expansion

$$\exp\left(\frac{\varepsilon}{y_0 + \varepsilon y_1 + \dots}\right) = 1 - \frac{\varepsilon}{y_0} + \frac{y_1}{y_0^2} \varepsilon^2 + \dots$$

15. Let $\theta = \theta_0 + \theta_1 \varepsilon + \dots$. Then, substituting, gives

$$\theta_0'' + \theta_1'' \varepsilon + \dots + \frac{1}{\varepsilon} (\varepsilon \theta_0 - \theta_1 \varepsilon^2 + (\theta_2 - \frac{1}{3} \theta_0) \varepsilon^3 + \dots) = 0.$$

Then $\theta_0'' + \theta_0 = 0$, $\theta_1'' + \theta_1 = 0$, $\theta_2'' + \theta_2 = \frac{1}{3} \theta_0$, and so on. The initial conditions give $\theta_0 = \cos \tau$ and $\theta_1 = 0$. Then

$$\theta_2'' + \theta_2 = -\frac{1}{3} \cos \tau.$$

For part (b), multiply the equation by θ' to get

$$\theta' \theta'' + \varepsilon^{-1} \sin(\varepsilon \theta) \theta' = 0.$$

This is the same as

$$\frac{1}{2} (\theta'^2)' - \frac{1}{\varepsilon^2} (\cos(\varepsilon \theta))' = 0.$$

Therefore

$$\frac{1}{2} \theta'^2 - \frac{1}{\varepsilon^2} \cos(\varepsilon \theta) = C.$$

From the initial conditions we get $C = -\frac{1}{\varepsilon^2} \cos \varepsilon$, and therefore the last equation can be written

$$\frac{\varepsilon}{\sqrt{2}} \frac{d\theta}{\sqrt{\cos(\varepsilon \theta) - \cos \varepsilon}} = \pm d\tau.$$

Now integrate over one-fourth of a period P to get

$$\frac{\varepsilon}{\sqrt{2}} \int_0^1 \frac{ds}{\sqrt{\cos(\varepsilon s) - \cos \varepsilon}} = \frac{P}{4}.$$

To get the expansion $P = 2\pi + \pi^2 \varepsilon^2 / 8 + \dots$, expand the integrand in powers of ε using the binomial theorem.

16. The three-term approximation is given by

$$y_a(t) = t + \frac{t^4}{12}\varepsilon + \frac{t^7}{504}\varepsilon^2.$$

The error is

$$E(t, \varepsilon) = y_a'' - \varepsilon t y_a = -\varepsilon^3 \frac{t^8}{504}.$$

Clearly the approximation is not uniform on $t \geq 0$.

17. The leading order solution is

$$y_0(t) = \sqrt{t}(1 - \ln t).$$

Substituting into the ODE gives

$$Ly_0 = -\frac{\varepsilon}{2}(1 + \ln t)t^{3/2}.$$

We find

$$|Ly_0| \leq 0.0448.$$

Thus one would expect y_0 to be a good approximation on $0 \leq t \leq e$.

18. Substitute the series $u = u_0 + u_1\varepsilon + \dots$ into the differential equation and initial conditions:

$$\begin{aligned} u_0' + u_1'\varepsilon + \dots + u_0 + u_1\varepsilon + \dots &= \frac{1}{1 + u_0\varepsilon + \dots} \\ &= 1 - u_0\varepsilon + 0(\varepsilon^2). \end{aligned}$$

To get the last step we used the geometric series expansion. Now collecting the coefficient of ε^0 we get the leading order problem

$$u_0' + u_0 = 1, \quad u_0(0) = 0.$$

Collecting the coefficients of ε we get

$$u_1' + u_1 = -u_0, \quad u_1(0) = 0.$$

The solution to the leading order problem (a linear equation) is

$$u_0 = 1 - e^{-t}.$$

The the next order problem becomes

$$u_1' + u_1 = e^{-t} - 1, \quad u_1(0) = 0.$$

This linear equation has solution

$$u_1 = (t + 1)e^{-t} - 1.$$

Therefore, a two-term approximation is

$$u(t) = 1 - e^{-t} + \varepsilon((t + 1)e^{-t} - 1) + \dots .$$

19. Let s be the speed of the wildebeest and $\sigma = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{dx}{dt} \sqrt{1 + y'^2}$ be the speed of the lion. The velocity of the lion is

$$\left(\frac{dy}{dx}, \frac{dy}{dt}\right) = \frac{\sigma}{\sqrt{(a-x)^2 + (b+st-y)^2}} (a-x, b+st-y).$$

Now,

$$y' = \frac{b+st-y}{a-x}, \quad y' = \frac{dy}{dx}.$$

Therefore,

$$\frac{d}{dt} y' = y'' \frac{dx}{dt} = \frac{(a-x)\left(s - \frac{dy}{dt}\right) + \frac{dx}{dt}(b+st-y)}{(a-x)^2}.$$

This simplifies to

$$\begin{aligned} (a-x)y'' &= \frac{s - \frac{dy}{dt}}{\frac{dx}{dt}} + \frac{dy}{dx} \\ &= \frac{s}{\frac{dx}{dt}} = \frac{s}{\sigma} \sqrt{1 + y'^2}. \end{aligned}$$

So we have

$$(a-x)y'' = \varepsilon \sqrt{1 + y'^2}, \quad y(0) = 0, \quad y'(0) = \frac{b}{a}.$$

Assuming

$$y = y_0(x) + \varepsilon y_1(x) + \dots,$$

the leading order problem is

$$(a-x)y_0'' = 0, \quad y_0(0) = 0, \quad y_0'(0) = \frac{b}{a},$$

which has solution

$$y_0(x) = \frac{b}{a}x.$$

At next order

$$(a-x)y_1'' = 1, \quad y_1(0) = 0, \quad y_1'(0) = 0.$$

Now,

$$y_1' = -\ln(a-x) + C.$$

Using the second initial condition, $C = \ln a$. Now integrate again to get

$$y_1(x) = -\int_0^x \ln(a-\xi) d\xi + x \ln a.$$

Then

$$y(x) = \frac{b}{a}x + \left(-\int_0^x \ln(a-\xi) d\xi + x \ln a\right) \varepsilon + \dots$$

2.2 Singular Perturbation

Exercises, page 111

1. (a) Consider the equation

$$\varepsilon x^4 + \varepsilon x^3 - x^2 + 2x - 1 = 0$$

If $x = O(1)$ then $x^2 - 2x + 1 \sim 0$ which means $x \sim 1, 1$, a double root near one. To find the remaining roots assume a dominant balance $\varepsilon x^4 \sim x^2$ with the remaining terms small. Then $x = O(1/\sqrt{\varepsilon})$. This is a consistent balance because $\varepsilon x^4, x^2 = O(1/\varepsilon)$ and the remaining three terms are order one and small in comparison. So the dominant balance is

$$\varepsilon x^4 - x^2 \sim 0,$$

which gives $x \sim \pm 1/\sqrt{\varepsilon}$. So the leading order roots are

$$1, 1, \pm 1/\sqrt{\varepsilon}.$$

- (b) The equation

$$\varepsilon x^3 + x - 2 = 0$$

has an order one root $x \sim 2$. The consistent dominant balance is $\varepsilon x^3 \sim x$ which gives $x = O(1/\sqrt{\varepsilon})$. In this case we have $\varepsilon x^3 + x \sim 0$, which gives the two other leading order roots as

$$x \sim \pm \frac{i}{\sqrt{\varepsilon}}$$

To find a higher order approximation for the root near $x = 2$ substitute $x = 2 + x_1\varepsilon + \dots$ into the equation and collect coefficients of ε to get $x_1 = -8$. Thus

$$x = 2 - 8\varepsilon + \dots$$

Since the other two roots are order $O(1/\sqrt{\varepsilon})$, let us choose a new order one variable y given by

$$y = x/(1/\sqrt{\varepsilon})$$

So, we are rescaling. Then the equation becomes

$$y^3 + y - 2\sqrt{\varepsilon} = 0$$

and the small term appears where it should in the equation. Now assume $y = y_0 + y_1\sqrt{\varepsilon} + \dots$, substitute into the equation, and collect coefficients to get at leading order

$$y_0^3 + y_0 = 0,$$

which gives $y_0 = \pm i$. At order $O(\sqrt{\varepsilon})$ we get the equation

$$3y_0^2 y_1 + y_1 - 2 = 0.$$

Thus $y_1 = -1$ and we have the expansions

$$y = \pm i - \sqrt{\varepsilon} + \dots$$

In terms of x ,

$$x = \pm \frac{i}{\sqrt{\varepsilon}} - 1 + \dots$$

(c) The equation

$$\varepsilon^2 x^6 - \varepsilon x^4 - x^3 + 8 = 0$$

has three order one roots as solutions of $-x^3 + 8 = 0$, or

$$x \sim 2, \quad -1 \pm \sqrt{3}i.$$

To find the other roots, the dominant balance is $\varepsilon^2 x^6 - x^3 \sim 0$, which gives

$$x \sim \frac{1}{\varepsilon^{2/3}} e^{2\pi i/3}, \quad \frac{1}{\varepsilon^{2/3}} e^{-2\pi i/3}.$$

(d) The equation

$$\varepsilon x^5 + x^3 - 1 = 0$$

has three order one roots (the cube roots of one) given by

$$x \sim 1, \quad -\frac{1}{2} \pm \sqrt{3}i.$$

The dominant balance for the remaining roots is between the first two terms which gives

$$x \sim \pm \frac{i}{\sqrt{\varepsilon}}.$$

2. Follow the given hint.

3. Observe that the equation can be written as a quadratic in ε :

$$2\varepsilon^2 + x\varepsilon + x^3 = 0.$$

Thus

$$\varepsilon = \frac{1}{4}(-x \pm \sqrt{x^2 - 8x^3}).$$

These two branches can be graphed on a calculator, and the graph shows that there is just one negative value for x , near $x = 0$, in the case that ε is small and positive. Thus assume

$$x = x_1 \varepsilon + x_2 \varepsilon^2 + \dots$$

Substituting into the given equation gives

$$x = -2\varepsilon + 8\varepsilon^2 + \dots$$

4. The problem is

$$\varepsilon y'' + y' + y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

When $\varepsilon = 0$ we get $y' + y = 0$ which has solution $y = ce^{-t}$. This cannot satisfy both boundary conditions so regular perturbation fails.

The characteristic equation is $\varepsilon m^2 + m + 1 = 0$ which has two real, negative roots given by

$$\begin{aligned} m_1 &= \frac{1}{2\varepsilon}(-1 + \sqrt{1 - 4\varepsilon}) = -1 + O(\varepsilon), \\ m_2 &= \frac{1}{2\varepsilon}(-1 - \sqrt{1 - 4\varepsilon}) = -\frac{1}{\varepsilon} + O(1). \end{aligned}$$

Here we have used the binomial expansion $\sqrt{1+x} = 1 + x/2 + O(x^2)$ for small x . Note that one of the roots is order one, and one of the roots is large. So the general solution is

$$y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}.$$

Applying the boundary conditions gives the exact solution

$$y(t) = \frac{e^{m_1 t} - e^{m_2 t}}{e^{m_1} - e^{m_2}}.$$

Sketches of this solution show a boundary layer near $t = 0$ where there is a rapid increase in $y(t)$. Observe that $e^{m_1} \gg e^{m_2}$. If $t = O(1)$ then

$$e^{m_1 t} \sim e^{-t}, \quad e^{m_2 t} \sim 0,$$

and thus

$$y(t) \sim \frac{e^{m_1 t}}{e^{m_1}} \sim e^{1-t},$$

which is an outer approximation. If $t = O(\varepsilon)$, then

$$\begin{aligned} y(t) &\sim \frac{e^{(-1+O(\varepsilon))t} - e^{(-1/\varepsilon+O(1))t}}{e^{m_1}} \\ &\sim \frac{e^{O(\varepsilon)} - e^{O(\varepsilon)} e^{-t/\varepsilon}}{e^{m_1}} \\ &\sim \frac{1 - e^{-t/\varepsilon}}{e^{-1}}. \end{aligned}$$

This is an inner approximation near $t = 0$.

2.3 Boundary Layer Analysis

Exercises, page 121

1. (a) Consider

$$\varepsilon y'' + 2y' + y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

By Theorem 2.9 there is a boundary layer at $x = 0$. Setting $\varepsilon = 0$ gives the outer approximation $y(x) = Ce^{-x/2}$. Use the right boundary condition to find $C = \sqrt{e}$. Then the outer approximation is

$$y_0(x) = \sqrt{e}e^{-x/2} = e^{(1-x)/2}.$$

The thickness of the layer is $\delta(\varepsilon) = \varepsilon$ and the inner equation is

$$Y'' + 2Y' + \varepsilon Y = 0.$$

The leading order inner approximation is

$$Y_i(\xi) = a + be^{-2\xi}, \quad \xi = \frac{x}{\varepsilon}.$$

Now, $Y_i(0) = a + b = 0$, and matching gives $a = \sqrt{e}$.

- (b) We have

$$\varepsilon y'' + y' + y^2 = 0, \quad y(0) = 1/4, \quad y(1) = 1/2.$$

There is an expected layer at $x = 0$. Setting $\varepsilon = 0$ and solving gives the outer solution

$$y_0(x) = \frac{1}{x+2}.$$

We applied the right boundary condition. In the boundary layer set $\xi = x/\varepsilon$ and $Y(\xi) = y(x)$. Then the inner equation is

$$Y'' + Y' + \varepsilon Y^2 = 0.$$

To leading order we have $Y_i'' + Y_i' = 0$ with $Y_i(0) = 1/4$. So the inner approximation is

$$Y_i(\xi) = A(1 - e^{-\xi}).$$

Matching gives $A = 1/2$ and so the uniform approximation is

$$y(x) = \frac{1}{x+2} - \frac{1}{4}e^{-x/\varepsilon}.$$

- (c) We have

$$\varepsilon y'' + (1+x)y' = 1, \quad y(0) = 0, \quad y(1) = 1 + \ln 2.$$

The outer solution is

$$y_0(x) = 1 + \ln(x+1).$$

The inner equation, with $\xi = x/\delta\varepsilon$, is

$$\frac{\varepsilon}{\delta^2} + (1 + \xi\delta)\frac{1}{\delta}Y' = 1.$$

Balancing gives $\delta = \varepsilon$ and the inner first-order approximate equation is

$$Y_i'' + Y_i' = 0, \quad Y_i(0) = 0.$$

So the inner solution is

$$Y_i(\xi) = A(1 - e^{-\xi}).$$

Matching gives $A = 1$ and so the uniform approximation is

$$y(x) = \ln(x+1) - e^{-x/\varepsilon}.$$

(d) The problem is

$$\varepsilon y'' + (1+t)y' + y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

By Theorem 3.1 in the text, there is a boundary layer at zero. The outer solution is $y_0 = 2/(t+1)$. In the boundary layer set $\tau = t/\varepsilon$ and $Y(\tau) = y(t)$. Then the inner equation is

$$Y'' + \varepsilon\tau + Y' + \varepsilon Y = 0.$$

To leading order we have $Y_i'' + Y_i' = 0$ with $Y_i(0) = 0$. So the inner approximation is

$$Y_i(\tau) = A(1 - e^{-\tau}).$$

Matching gives $A = 2$ and so the uniform approximation is

$$y(t) = \frac{2}{t+1} + 2(1 - e^{-t/\varepsilon}) - 2.$$

(e) The problem is

$$\varepsilon y'' + t^{1/3}y' + y = 0, \quad y(0) = 0, \quad y(1) = e^{-3/2}.$$

There is a boundary layer near $t = 0$. The outer solution is

$$y_0(t) = \exp(-1.5t^{2/3}).$$

In the inner region set $\tau = t/\delta(\varepsilon)$. Then the dominant balance is between the first and second terms and $\delta = \varepsilon^{3/4}$. So the inner approximation to leading order is

$$Y_i'' + \tau^{1/3}Y_i' = 0.$$

Solving gives

$$Y_i(\tau) = c \int_0^\tau \exp(-0.75s^{4/3}) ds.$$

Pick an intermediate variable to be $\eta = t/\sqrt{\varepsilon}$. Then matching gives

$$c = \left(\int_0^\infty \exp(-0.75s^{4/3}) ds \right)^{-1}$$

(f) Consider

$$\varepsilon y'' + xy' - xy = 0, \quad y'(0) = 1, \quad y(1) = e.$$

By Theorem 2.9 there is a layer at $x = 0$. The outer approximation is

$$y_0(x) = e^{1-x}.$$

Letting $\xi = x/\delta(\varepsilon)$ in the layer, we find from dominant balance that $\delta(\varepsilon) = \sqrt{\varepsilon}$. The inner equation is

$$Y_i'' + \xi Y_i' = 0.$$

Then

$$Y_i(\xi) = a \int_0^\xi e^{s^2/2} ds + b.$$

Then $Y_i(0) = b = 0$. Matching gives

$$a = e \left(\int_0^\infty e ds \right)^{-1}.$$

(g) The problem is

$$\varepsilon y'' + 2y' + e^y = 0, \quad y(0) = y(1) = 0$$

There is a layer at zero. The outer solution is

$$y_0(t) = -\ln \frac{t+1}{2}$$

In the boundary layer the first two terms dominate and $\delta(\varepsilon) = \varepsilon$. The inner solution is

$$Y_i(\tau) = A(1 - e^{-2\tau})$$

Matching gives $A = \ln 2$. The uniform approximation is

$$y(t) = \ln 2(1 - e^{-2t/\varepsilon}) - \ln \frac{t+1}{2} - \ln 2.$$

(h) The problem is

$$\varepsilon y'' - (2 - t^2)y = -1, \quad y(-1) = y(1) = 1$$

Now there are two layers near $t = -1$ and $t = 1$. The outer solution, which is valid in the interval $(-1, 1)$, away from the layers is

$$y_0(t) = \frac{1}{2 - t^2}$$

In the layer near $t = 1$ set $\tau = (1 - t)/\delta(\varepsilon)$. We find $\delta = \sqrt{\varepsilon}$ with inner equation, to leading order, $Y_i'' - Y_i = 1$. The inner solution is

$$Y_i(\tau) = 1 + ae^{-\tau} - (1 + a)e^\tau$$

In the layer near $t = -1$ set $\tau = (t - 1)/\delta(\varepsilon)$. We find $\delta = \sqrt{\varepsilon}$ with inner equation, to leading order, $(Y_i^*)'' - Y_i^* = -1$. The inner solution is

$$Y_i^*(\tau) = 1 + be^{-\tau} - (1 + b)e^\tau$$

Matching gives $a = b = 1$ and the uniform approximation is

$$y(t) = \frac{1}{2 - t^2} - e^{(t-1)/\sqrt{\varepsilon}} - e^{(t+1)/\sqrt{\varepsilon}}$$

(i) The problem is

$$\varepsilon y'' - b(x)y' = 0, \quad y(0) = \alpha, \quad y(1) = \beta.$$

There is an expected layer at $x=1$ because the coefficient of y' is negative. Therefore the outer solution is

$$y_0(x) = \text{const.} = \alpha.$$

Now make the change of variables

$$\xi = \frac{1 - x}{\delta(\varepsilon)}.$$

Then $x = 1 - \xi\delta$ and the differential equation becomes

$$\frac{\varepsilon}{\delta^2} Y'' + [b(1) - b'(1)\xi\delta + \dots] \frac{1}{\delta} Y' = 0.$$

Balancing terms gives $\delta = \varepsilon$ and the leading order inner equation is

$$Y_i'' + b(1)Y_i' = 0.$$

We have $Y_i(0) = \beta$. The solution is

$$Y_i(\xi) = A + (\beta - A)e^{-b(1)\xi}.$$

Matching gives $A = \alpha$. Then, a uniform approximation is

$$y(x) = \alpha + (\beta - \alpha)e^{-b(1)x/\varepsilon}.$$

(j) Consider

$$\varepsilon y'' - 4(\pi - x^2)y = \cos x, \quad y(0) = 0, \quad y(\pi/2) = 1.$$

2. The solution is

$$u(x) = a \sin(x/\sqrt{\varepsilon}) + b \cos(x/\sqrt{\varepsilon})$$

where a and b are determined uniquely by the boundary conditions. This is a very rapidly oscillating function over the entire interval. To apply perturbation methods we set $\varepsilon = 0$ to get the outer solution $u(x) = 0$. This constant solution cannot be matched to rapid oscillations.

3. See problem 1(g).

4. Consider

$$\varepsilon u'' - (2x + 1)u' + 2u = 0, \quad u(0) = 1, \quad u(1) = 0.$$

5. The problem is

$$\varepsilon y'' + \frac{1}{x}y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

which is an initial value problem and appears to be singular. But, the outer solution is

$$y_0(x) = Ce^{-x^2/2}$$

and it is observed that it satisfies both initial conditions when $C = 1$. It also satisfies the ODE uniformly, i.e.,

$$\varepsilon y_0'' + \frac{1}{x}y_0' + y_0 = \varepsilon(x^2 - 1)e^{-x^2/2} = O(\varepsilon)$$

Thus it provides a uniform approximation and the problem does not have a layer. It is instructive to try to put a layer at $x = 0$; one finds that no scaling is possible.

6. Consider

$$\varepsilon y'' + \left(x - \frac{1}{2}\right)y = 0, \quad y(0) = 1, \quad y(1) = 2.$$

There is a layer at both $x = 0$ and $x = 1$. The outer solution is $y_0(x) = 0$. By dominant balancing, the width of the layer in both cases is $\delta(\varepsilon) = \sqrt{\varepsilon}$. The inner equation at $x = 0$ is

$$Y_l'' - \frac{1}{2}Y_l = 0,$$

where

$$\xi = \frac{x}{\sqrt{\varepsilon}}.$$

The inner equation at $x = 1$ is

$$Y_r'' - \frac{1}{2}Y_r = 0,$$

where

$$\xi = \frac{1-x}{\sqrt{\varepsilon}}.$$

. Using the appropriate boundary conditions gives the left and right inner solutions

$$Y_l(\xi) = e^{-\xi/\sqrt{2}}, \quad Y_r(\xi) = e^{-\xi/\sqrt{2}}.$$

The uniform approximation is

$$y(x) = e^{-x/\sqrt{2\varepsilon}} + e^{(1-x)/\sqrt{2\varepsilon}}.$$

7. There is a layer at $x = 0$. The outer solution is $y_0(x) = -\exp(-x)$. The layer has width $\delta(\varepsilon) = \varepsilon$ and the leading order inner problem is

$$Y_i'' + Y_i' = 0.$$

It has solution $Y_i(\xi) = A + B \exp(-\xi)$. The boundary condition gives $A + B = 1$. Thus $Y_i(\xi) = A + (1-A) \exp(-\xi)$. Matching give $A=-1$ and the uniform approximation is

$$y(x) = 2e^{-x/\varepsilon} - e^{-x}.$$

8. Assuming a layer at $x = 0$ we obtain outer solution

$$y_0(x) = f'(x) - f'(1).$$

Now assume $\xi = x/\delta$. In transforming the equation, we need

$$\begin{aligned} \frac{1}{f'(\xi\delta)} &= \frac{1}{f'(0) + f''(0)\xi\delta + \dots} \\ &= \frac{1}{f'(0)} - f''(0)\xi\delta + \dots \end{aligned}$$

The dominant balance is $\delta = \varepsilon$ and the leading order approximation is

$$Y'' - \frac{1}{f'(0)}Y = 0,$$

giving

$$Y(\xi) = A + B e^{\xi/f'(0)}.$$

Here we need $f'(0) = b < 0$. Matching gives $A = f(0) - f(1)$.

9. Note that the interval should be $0 < x < b$, and not $0 < x < 1$. The outer solution is

$$u_0(x) = \frac{1}{\cos(b-x)},$$

which satisfies the right boundary condition $u'(b) = 0$ automatically. We try a layer at $x = 0$ by defining the scaling

$$\xi = x/\delta(\varepsilon).$$

Then the equation becomes

$$-\frac{\varepsilon}{\delta}Y'' + \cos(b - \xi\delta)U = 1.$$

We have to expand everything in terms of δ . We have

$$\begin{aligned}\cos(b - x) &= \cos b \cos(\xi\delta) + \sin b \sin(\xi\delta) \\ &= \cos b \left(1 - \frac{1}{2}(\xi\delta)^2 + \dots\right) + \sin b \left(\xi\delta - \frac{1}{6}(\xi\delta)^3 + \dots\right).\end{aligned}$$

Substituting into the differential equation leads to a dominant balance giving

$$\delta = \sqrt{\varepsilon}.$$

The leading order inner equation is

$$-U'' + (\cos b)U = 1.$$

The general solution is

$$U(\xi) = Ae^{-(\cos b)\xi} + Be^{(\cos b)\xi} + \frac{1}{\cos b}.$$

Now, $U(0) = A + B + \frac{1}{\cos b} = 0$. Thus $B = -\left(A + \frac{1}{\cos b}\right)$. For matching to work we must have $A = -\frac{1}{\cos b}$. Then we have the uniform approximation

$$\begin{aligned}u(x) &= -\frac{1}{\cos b}e^{-(\cos b)x/\sqrt{\varepsilon}} + \frac{1}{\cos b} + \frac{1}{\cos(b-x)} - \frac{1}{\cos b} \\ &= -\frac{1}{\cos b}e^{-(\cos b)x/\sqrt{\varepsilon}} + \frac{1}{\cos(b-x)}.\end{aligned}$$

10. The outer solution is clearly

$$u_0(x) = 0,$$

with a layer at $x = 1$. (Theorem 2.9 applies.) In the layer,

$$\xi = \frac{1-x}{\varepsilon}.$$

Then the differential equation transforms to

$$\frac{1}{\varepsilon}U'' - a(1 - \xi\varepsilon)\frac{-1}{\varepsilon}U' = f(x).$$

Expanding,

$$U'' + (a(1) - a'(1)\xi\varepsilon + \dots)U' = \varepsilon f(x).$$

To leading order,

$$U_i'' + (a(1) - a'(1)\xi\varepsilon + \dots)U_i' = 0,$$

which gives

$$U_i(\xi) = A + Be^{-a(1)\xi}.$$

The boundary condition is $U_i(0) = A + Be^{-a(1)} = -f(1)/a(1)$.

11. See problem 12e.

12. (a) By Theorem 3.1 the problem

$$\varepsilon y'' + (1 + x^2)y' - x^3y = 0, \quad y(0) = y(1) = 1$$

has a layer of order $\delta = \varepsilon$ at $t = 0$. The outer solution is

$$y_0(x) = \sqrt{\frac{2}{1+x^2}} \exp\left(\frac{x^2-1}{2}\right)$$

The inner solution is

$$Y_i(\xi) = A + (1-A)e^{-\xi}$$

Matching gives $A = \sqrt{2/e}$.

(b) By Theorem 3.1 the problem

$$\varepsilon y'' + (\cosh t)y - y = 0, \quad y(0) = y(1) = 1$$

has a layer of width $\delta = \varepsilon$ near $t = 0$. The outer solution is

$$y_0(t) = \exp(2 \arctan e^t - 2 \arctan e)$$

In the layer use the expansion

$$\cosh z = 1 + \frac{1}{2}z^2 + \dots$$

The inner approximation is

$$Y_i(\tau) = 1 - A + Ae^{-\tau}$$

Matching gives $A = 1 - \exp(\pi/2 - 2 \arctan e)$.

(c) The problem

$$\varepsilon y'' + \frac{2\varepsilon}{t}y' - y = 0, \quad y(0) = 0, \quad y'(1) = 1$$

If we assume a layer at $t = 1$ the outer solution is $y_0(t) = 0$. In the inner region near $t = 1$ the dominant balance is between the first and last terms and the width of the layer is $\delta = \sqrt{\varepsilon}$. The inner variable is $\tau = (1-t)/\delta$. The inner solution is

$$Y_i(\tau) = (\sqrt{\varepsilon} + b)e^{-\tau} + be^{\tau}$$

We must set $b = 0$ to stay bounded. So the uniform solution is

$$y(t) = \sqrt{\varepsilon} e^{(t-1)/\sqrt{\varepsilon}}$$

- (d) In this problem we have an outer solution $y_0(t) = 0$ which satisfies both boundary conditions exactly. So we have an exact solution $y \equiv 0$.
- (e) The problem is

$$\varepsilon y'' + \frac{1}{x+1} y' + \varepsilon y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

The outer approximation is $y_0(t) = \text{const.}$, so there are several possibilities. But, because the y' coefficient is positive, we suspect the layer is at $x = 0$. Therefore we apply the right boundary condition to the outer solution, giving $y_0(x) = 1$. Then assume a layer at $x = 0$ of width $\delta(\varepsilon)$; i.e.,

$$\xi = \frac{x}{\delta(\varepsilon)}.$$

The inner problem is

$$\frac{\varepsilon}{\delta(\varepsilon)^2} Y'' + \frac{1}{(1 + \xi\delta(\varepsilon))\delta(\varepsilon)} Y' + \varepsilon Y = 0.$$

Expanding the second term in a geometric series gives

$$\frac{\varepsilon}{\delta(\varepsilon)^2} Y'' + (1 - \delta(\varepsilon)\xi + \dots) \frac{1}{\delta(\varepsilon)} Y' + \varepsilon Y = 0.$$

Balancing gives $\delta(\varepsilon) = \varepsilon$ and the leading order equation is

$$Y_i'' + Y_i = 0,$$

which gives

$$Y_i''(\xi) = a(1 - e^{-\xi}).$$

Matching gives $a = 1$. Therefore a uniform approximation is

$$y_u(x) = 1 + 1 - e^{-x/\varepsilon} - 1 = 1 - e^{-x/\varepsilon}.$$

This approximation satisfies $y_u(0) = 0$, $y_u(1) = 1 - \exp(-1/\varepsilon)$, which is one minus an exponentially small term. Substituting into the differential equation gives

$$\varepsilon y_u'' + \frac{1}{x+1} y_u' + \varepsilon y_u = \varepsilon + \text{exponentially small term.}$$

Boundary layer at the right boundary. Just as illustration we show how to proceed if the differential equation is

$$\varepsilon y'' - \frac{1}{x+1} y' + \varepsilon y = 0$$

with the same boundary conditions. Now the y' coefficient is negative and we expect a layer at $x = 1$. Therefore the outer solution is $y_0(x) = 0$ (from applying the left boundary condition). Now let

$$\xi = \frac{1-x}{\delta(\varepsilon)}, \quad (\text{note the change})$$

Then the inner problem is

$$\frac{\varepsilon}{\delta(\varepsilon)^2} Y'' - (1 - \delta(\varepsilon)\xi + \dots) \frac{-1}{\delta(\varepsilon)} Y' + \varepsilon Y = 0.$$

Note the minus sign that appears when transforming y' . Then, dominant balance forces $\delta(\varepsilon) = \varepsilon$ and the leading order equation is

$$Y_i'' + Y_i = 0, \quad Y_i(0) = 1.$$

The inner solution is

$$Y_i''(\xi) = a + be^{-\xi}.$$

Applying the boundary condition gives $a + b = 1$, so

$$Y_i''(\xi) = a + (1-a)e^{-\xi}.$$

Matching gives $a = 0$ because $y_0(x) \rightarrow 0$ as $x \rightarrow 1$. Therefore a uniform approximation is outer + inner - common limit, or

$$y_u(x) = e^{-(1-x)/\varepsilon}.$$

2.4 Initial Layers

Exercises, page 133

1. The equation is

$$\varepsilon y' + y = e^{-t}, \quad y(0) = 2.$$

Set $\varepsilon = 0$ to obtain the outer solution $y_0(t) = e^{-t}$, away from $t = 0$. Rescale near zero via $\tau = t/\delta(\varepsilon)$. Then, in the usual way, we find $\delta(\varepsilon) = \varepsilon$ and the leading order inner problem is

$$Y_i' + Y_i = 1.$$

This has solution $Y_i(\tau) = 1 + Ce^{-\tau}$. Applying the initial condition gives $C = 1$. We find the matching condition holds automatically. So the uniform approximation is

$$y(t) = e^{-t} + e^{-t/\varepsilon}.$$

2. The equation is

$$\varepsilon y'' + b(t)y' + y = 0, \quad y(0) = 1, \quad y'(0) = -\frac{\beta}{\varepsilon} + \gamma.$$

The outer solution is

$$y_0(t) = C \exp\left(-\int_0^t b(z)^{-1} dz\right).$$

Near $t = 0$ set $\tau = \delta(\varepsilon)$. Then

$$\frac{\varepsilon}{\delta(\varepsilon)^2} Y'' + \frac{1}{\delta(\varepsilon)} (b(0) + b'(0)\tau\delta(\varepsilon) + \dots) Y' + Y = 0.$$

The dominant balance gives $\delta(\varepsilon) = \varepsilon$ and the inner problem, to leading order is

$$Y_i'' + b(0)Y_i' = 0.$$

This has general solution $Y_i = A + Be^{-b(0)\tau}$. From $Y_i(0) = 1$ we get $A + B = 1$. The other initial condition leads to $Y_i'(0) = \beta$. Therefore the inner approximation in the initial layer is

$$Y_i(\tau) = \left(1 + \frac{\beta}{b(0)}\right) - \frac{\beta}{b(0)} e^{-b(0)\tau}.$$

The matching condition gives $C = 1 + \frac{\beta}{b(0)}$. So, a uniform approximation is

$$y(t) = \left(1 + \frac{\beta}{b(0)}\right) \exp\left(-\int_0^t b(z)^{-1} dz\right) - \frac{\beta}{b(0)} e^{-b(0)t/\varepsilon}.$$

3. The problem is

$$\varepsilon y'' + (t+1)^2 y' = 1, \quad y(0) = 1, \quad \varepsilon y'(0) = 1.$$

The outer approximation is

$$y_0(t) = -\frac{1}{t+1} + C.$$

Rescaling in the initial layer gives

$$\frac{\varepsilon}{\delta(\varepsilon)^2} Y'' + \frac{1}{\delta(\varepsilon)} (1 + \tau\delta(\varepsilon))^2 Y' = 1.$$

The dominant balance is $\delta = \varepsilon$ and to leading order we have

$$Y'' + Y' = 0,$$

which gives

$$Y_i(\tau) = A + Be^{-\tau}.$$

Now, $Y(0) = 1$ gives $A + B = 1$. And, the other initial condition gives $Y'(0) = 1$. So the inner solution is

$$Y_i(t) = 2 - e^{-\tau}.$$

Matching gives $C=3$. Then the uniform approximation is

$$y(t) = -e^{-t/\varepsilon} + 3 - \frac{1}{t+1}.$$

4. The damped oscillator is governed by

$$my'' + ay' + ky^3 = 0, \quad y(0) = 0, \quad my'(0) = I.$$

Let $\tau = t/(a/k)$ and $u = y/(I/a)$ to get

$$\varepsilon u'' + Y' + u' + u = 0, \quad u(0) = 0, \quad \varepsilon u'(0) = 1.$$

The standard singular perturbation method with an initial layer at $t = 0$ leads to the approximation

$$u(\tau) = e^{-\tau} - e^{-\tau/\varepsilon}.$$

5. (a) In this case the system is

$$x' = y - \varepsilon \sin x, \quad \varepsilon y' = x^2 y + \varepsilon y^3$$

with initial conditions $x(0) = k$, $y(0) = 0$. Setting $\varepsilon = 0$ we get the outer equations

$$x'_0 = y_0, \quad 0 = x_0^2 y_0$$

Here we can choose $y_0 = 0$ and $x_0 = k$ and the initial conditions are met. So this problem does not have an initial layer. It is a regular perturbation problem with leading order solution

$$x_0(t) = k, \quad y_0(t) = 0$$

It is instructive for the student to assume a layer near $t = 0$ and carry out the analysis to find that the inner approximation agrees with the outer approximation.

(b) The problem is

$$u' = v, \quad \varepsilon v' = u^2 - v, \quad u(0) = 1, \quad v(0) = 0$$

Assume a boundary layer near $t = 0$. Then the outer problem is

$$u'_0 = v_0, \quad v_0 = -u_0^2$$

Then

$$u'_0 = -u_0^2$$

which has solution (separate variables)

$$u_0(t) = \frac{1}{t+c}, \quad v_0(t) = \frac{-1}{(t+c)^2}$$

In the boundary layer take $\eta = t/\varepsilon$. Then the inner problem is

$$U' = \varepsilon V, \quad V' = -U^2 - V, \quad U(0) = 1, \quad V(0) = 0$$

Thus, setting $\varepsilon = 0$ and solving yields the inner approximation

$$U(\eta) = \text{const.} = 1, \quad V(\eta) = e^{-\eta} - 1$$

Matching gives

$$\lim_{t \rightarrow 0} u_0(t) = \lim_{\eta \rightarrow \infty} U(\eta)$$

or $1/c = 1$. Hence, $c = 1$. Then the uniform approximation is

$$u = \frac{1}{t+1} + 1 - 1 = \frac{1}{t+1}$$

and

$$v = \frac{-1}{(t+1)^2} + e^{-t/\varepsilon} - 1 - (-1) = \frac{-1}{(t+1)^2} + e^{-t/\varepsilon}$$

6. The governing equations are

$$a' = -k_f a + k_b b, \quad b' = k_f a - k_b b.$$

Therefore $a + b$ is constant, and so $a + b = a_0$, giving $b = a_0 - a$. Then the a -equation becomes

$$a' = -(k_f + k_b)a + k_b a_0,$$

which has general solution

$$a(t) = C e^{-(k_b+k_f)t} + \frac{a_0 k_b}{k_b + k_f}.$$

The constant C can be determined from the initial condition.

7. The governing equations for the reaction $X + Y \rightarrow Z$ are

$$x' = -kxy, \quad y' = -kxy.$$

Therefore $x - y = C$, where C is constant. Hence,

$$x' = -kx(x - C) = kx(C - x),$$

which is the logistic equation.

8. The governing differential equation is

$$x' = -rk(T)x, \quad T = T_0 + h(x - x_0).$$

Making all the suggested changes of variables gives

$$\theta' = e^A e^{-A/\theta} (1 + \beta - \theta), \quad \theta(0) = 1,$$

where $A = E/RT_0$ and $\beta = -hx_0/T_0$. For small A take

$$\theta = \theta_0 + \theta_1 A + \theta_2 A^2 + \dots$$

To leading order

$$\theta'_0 = 1 + \beta\theta_0, \quad \theta_0(0) = 1,$$

which has solution

$$\theta_0(\tau) = (1 - 1/\beta)e^{\beta t} + \frac{1}{\beta}.$$

For large A take

$$\theta = \theta_0 + \theta_1 \frac{1}{A} + \theta_2 \frac{1}{A^2} + \dots$$

9.

2.5 WKB Approximation

Exercises, page 141

1. Letting $\varepsilon = 1/\sqrt{\lambda}$ we have

$$\varepsilon^2 y'' - (1 + x^2)^2 y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

This is the non-oscillatory case. From equation (2.96) of the text the WKB approximation is, after applying the condition $y(0) = 0$,

$$\begin{aligned} y_{WKB} &= \frac{c_1}{1 + x^2} \left[\exp \left(\sqrt{\lambda} \int_0^x (1 + \xi^2)^2 d\xi \right) - \exp \left(-\sqrt{\lambda} \int_0^x (1 + \xi^2)^2 d\xi \right) \right] \\ &= \frac{2c_1}{1 + x^2} \sinh \left(\sqrt{\lambda} \int_0^x (1 + \xi^2)^2 d\xi \right). \end{aligned}$$

Applying the condition $y'(0) = 1$ gives $c_1 = 1/2$.

2. Letting $\varepsilon = 1/\sqrt{\lambda}$ we have

$$y'' + \lambda(x + \pi)^4 y = 0, \quad y(0) = y(\pi) = 0.$$

This is the oscillatory case and the method in Example 2.15 applies. The large eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{\int_0^\pi (x + \pi)^2 dx} \right)^2 = \frac{9n^2}{49\pi^4}.$$

The eigenfunctions are

$$\begin{aligned} \frac{C}{x + \pi} \sin \left(\frac{3n}{7\pi^2} \int_0^x (\xi + \pi)^2 d\xi \right) \\ = \frac{C}{x + \pi} \sin \left(\frac{3n}{7\pi^2} \left(\frac{x^3}{3} + \pi x^2 + \pi^2 x \right) \right). \end{aligned}$$

3. Let $\lambda = 1/\varepsilon^2$ and rewrite the problem as

$$\varepsilon^2 y'' + xy = 0, \quad y(1) = y(4) = 0.$$

Proceed as in the oscillatory case.

4. Straightforward substitution.

5.

6. Make the change of variables $\tau = \varepsilon t$. Then the differential equation becomes

$$\varepsilon^2 \frac{d^2 y}{d\tau^2} + q(\tau)^2 y = 0.$$

We can think of the equation $y'' + q(\varepsilon t)^2 y = 0$ as a harmonic oscillator where the frequency is $q(\varepsilon t)$, which is time-dependent. If ε is small, it will take a large time t before there is significant changes in the frequency. Thinking of it differently, if $q(t)$ is a given frequency, then the graph of $q(\varepsilon t)$ will be stretched out; so the frequency will vary slowly.

7. Here we apply the ideas in Example 2.15 with $k(x) = e^{2x}$ and $\varepsilon = 1/\sqrt{\lambda}$. Then the WKB approximation is

$$y_{WKB} = \frac{c_1}{e^x} \sin \left(\frac{\sqrt{\lambda}}{2} (e^{2x} - 1) \right) + \frac{c_2}{e^x} \cos \left(\frac{\sqrt{\lambda}}{2} (e^{2x} - 1) \right).$$

Applying the condition $y(0) = 0$ gives $c_2 = 0$. Then

$$y_{WKB} = \frac{c_1}{e^x} \sin \left(\frac{\sqrt{\lambda}}{2} (e^{2x} - 1) \right)$$

Then $y(1) = 0$ gives

$$\sin \left(\frac{\sqrt{\lambda}}{2} (e - 1) \right) = 0,$$

and this forces

$$\lambda = \frac{4\pi^2}{(e - 1)^2}$$

for large n . [Note the typographical error—the boundary conditions should be homogeneous.]

8. Let $\varepsilon = 1/\lambda$ to obtain

$$\varepsilon^2 y'' + (x^2 + \varepsilon^2 x)y = 0.$$

Now let $y = \exp(iu/\varepsilon)$ and proceed as in the derivation of the WKB approximation.

2.6 Asymptotic Expansion of Integrals

Exercises, page 148

- 1.
2. Making the substitution $t = \tan^2 \theta$ gives

$$I(\lambda) = \int_0^{\pi/2} e^{-\lambda \tan^2 \theta} d\theta = \frac{1}{2} \int_0^\infty \frac{e^{-\lambda t} dt}{(1+t)\sqrt{t}}$$

Now Watson's lemma (Theorem 6.1) applies. But we proceed directly by expanding $1/(1+t)$ in its Taylor series

$$\frac{1}{1+t} = 1 - t + t^2 - \dots$$

which gives

$$I(\lambda) = \frac{1}{2} \int_0^\infty \frac{e^{-\lambda t}}{\sqrt{t}} (1 - t + t^2 - \dots) dt$$

Now let $u = \lambda t$. This gives

$$I(\lambda) = \frac{1}{2\sqrt{\lambda}} \int_0^\infty e^{-u} \left(\frac{1}{\sqrt{u}} - \frac{\sqrt{u}}{\lambda} + \frac{u^{3/2}}{\lambda^2} + \dots \right) du$$

Then, using the definition of the gamma function,

$$I(\lambda) = \frac{1}{2\sqrt{\lambda}} \left(\Gamma\left(\frac{1}{2}\right) - \frac{1}{\lambda} \Gamma\left(\frac{3}{2}\right) + \frac{1}{\lambda^2} \Gamma\left(\frac{5}{2}\right) + \dots \right)$$

3. Assume that g has a maximum at b with $g'(b) > 0$. Then expand

$$g(t) = g(b) + g'(b)(t-b) + \dots$$

The integral becomes

$$\begin{aligned} I(\lambda) &= \int_a^b f(t) e^{\lambda g(t)} dt \\ &= \int_a^b f(t) e^{\lambda(g(b) + g'(b)(t-b) + \dots)} dt \\ &\approx f(b) e^{\lambda g(b)} \int_a^b e^{\lambda g'(b)(t-b)} dt \end{aligned}$$

Now make the substitution $v = \lambda g'(b)(t - b)$ to obtain

$$I(\lambda) \approx f(b)e^{\lambda g(b)} \frac{1}{\lambda g'(b)} \int_{\lambda g'(b)(a-b)}^0 e^v dv$$

or

$$I(\lambda) \approx \frac{f(b)e^{\lambda g(b)}}{\lambda g'(b)} \int_{-\infty}^0 e^v dv$$

or

$$I(\lambda) \approx \frac{f(b)e^{\lambda g(b)}}{\lambda g'(b)}$$

for large λ .

If the maximum of g occurs at $t = a$ with $g'(a) < 0$, then it is the same calculation. We expand g in its Taylor series about $t = a$ and we obtain the same solution except for a minus sign and the b in the last formula replaced by a .

4. (a) We have, using a Taylor expansion,

$$\begin{aligned} I(\lambda) &= \int_0^\infty e^{-\lambda t} \ln(1+t^2) dt \\ &= \int_0^\infty e^{-\lambda t} \left(t^2 - \frac{t^4}{2} + \frac{t^6}{3} - \dots \right) dt \end{aligned}$$

Now let $u = \lambda t$ and we get

$$I(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{-u} \left(\frac{u^2}{\lambda^2} - \frac{u^4}{2\lambda^4} + \frac{u^6}{3\lambda^6} + \dots \right) dt$$

Using the definition of the gamma function, we obtain

$$I(\lambda) = \frac{1}{\lambda} \left(\frac{2!}{\lambda^2} - \frac{4!}{2\lambda^4} + \frac{6!}{3\lambda^6} + \dots \right)$$

- (b) Let $g(t) = 2t - t^2$. This function has its maximum at $t = 1$ where $g'(1) = 0$ and $g''(1) = -2$. Take $f(t) = \sqrt{1+t}$. Then

$$\begin{aligned} I(\lambda) &= \int_0^1 \sqrt{1+t} e^{\lambda(2t-t^2)} dt \approx \frac{1}{2} f(1) e^{\lambda g(1)} \sqrt{\frac{-2\pi}{\lambda g''(1)}} \\ &= \sqrt{\frac{\pi}{2\lambda}} e^\lambda \end{aligned}$$

- (c) Let $g(t) = 1/(1+t)$. This function has its maximum, with a negative derivative, at $t = 1$. Thus Exercise 6.3 holds. With $f(t) = \sqrt{3+t}$ we have

$$I(\lambda) \approx -\frac{f(1)e^{\lambda g(1)}}{\lambda g'(1)} = \frac{8}{\lambda} e^{\lambda/2}$$

5. We have

$$\Gamma(x+1) = \int_0^{\infty} u^x e^{-u} du$$

Integrate by parts by letting $r = u^x$ and $ds = e^{-u} du$. Then the integral becomes

$$\int_0^{\infty} u^x e^{-u} du = x \int_0^{\infty} e^{-u} u^{x-1} du = x\Gamma(x)$$

6.

7.

8.

9.

10.

11.

12. (b) Using the fact that $e^{-t} < 1$ for $t > 0$, we have

$$\begin{aligned} |r_n(\lambda)| &= n! \int_{\lambda}^{\infty} \frac{e^{-t}}{t^{n+1}} dt \\ &\leq n! \int_{\lambda}^{\infty} \frac{1}{t^{n+1}} dt \\ &= n! \frac{-1}{nt^n} \Big|_{\lambda}^{\infty} = (n-1)! \frac{1}{\lambda^n} \rightarrow 0 \end{aligned}$$

as $\lambda \rightarrow \infty$.

(c) Observe that

$$\int_{\lambda}^{\infty} \frac{e^{-t}}{t^{n+1}} dt \leq \frac{e^{-\lambda}}{\lambda^{n+1}}$$

Then the ratio of r_n to the last term of the expansion is

$$\frac{|r_n(\lambda)|}{(n-1)!e^{-\lambda}/\lambda^n} \leq \frac{n}{\lambda}$$

This tends to zero as $\lambda \rightarrow \infty$. So the remainder is little oh of the last term, and so we have an asymptotic series.

(d) Fix λ . The n th term of the series is does not converge to zero as $n \rightarrow \infty$. Therefore the series does not converge.

13. We have

$$I(\lambda) = \int_0^{\infty} \frac{1}{(t+\lambda)^2} e^{-t} dt$$

We integrate by parts letting

$$u = \frac{1}{(t+\lambda)^2}, \quad dv = e^{-t}$$

We get

$$I(\lambda) = \frac{1}{\lambda^2} - 2 \int_0^{\infty} \frac{1}{(t+\lambda)^3} e^{-t} dt$$

Now integrate by parts again via

$$u = \frac{1}{(t+\lambda)^3}, \quad dv = e^{-t}$$

Then

$$I(\lambda) = \frac{1}{\lambda^2} - \frac{2}{\lambda^3} + 6 \int_0^{\infty} \frac{1}{(t+\lambda)^4} e^{-t} dt$$

Continuing in the same manner gives

$$\begin{aligned} I(\lambda) &= \frac{1}{\lambda^2} - \frac{2!}{\lambda^3} + \frac{3!}{\lambda^4} \\ &+ \cdots + \frac{n!}{\lambda^{n+1}} (-1)^{n+1} + (n+1)! \int_0^{\infty} \frac{1}{(t+\lambda)^{n+2}} e^{-t} dt \end{aligned}$$

Chapter 3

Calculus of Variations

3.1 Variational Problems

Exercises page 158

1. The functional is

$$J(y) = \int_0^1 (y' \sin \pi y - (y+t)^2) dt.$$

First note that if $y(t) = -t$ then $J(y) = 2/\pi$. Now we have to show that $J(y) < 2/\pi$ for any other y . To this end,

$$\begin{aligned} J(y) &= \int_0^1 (y' \sin \pi y - (y+t)^2) dt \\ &\leq \int_0^1 y' \sin \pi y dt \\ &= -\frac{1}{\pi} \int_0^1 (\cos \pi y)' dt \\ &= -\frac{1}{\pi} (\cos \pi y(1) - \cos \pi y(0)) \leq \frac{2}{\pi}. \end{aligned}$$

2. Hint: substitute

$$y(x) = x + c_1 x(1-x) + c_2 x^2(1-x)$$

into the functional $J(y)$ to obtain a function $F = F(c_1, c_2)$ of the two variables c_1 and c_2 . Then apply ordinary calculus techniques to F to find the values that minimize F , and hence J . That is, set $\nabla F = 0$ and solve for c_1 and c_2 .

3.2 Necessary Conditions for Extrema

Exercises, page 166

- (a) The set of polynomials of degree ≤ 2 is a linear space. (b) The set of continuous functions on $[0, 1]$ satisfying $f(0) = 0$ is a linear space. (c) The set of continuous functions on $[0, 1]$ satisfying $f(0) = 1$ is not a linear space because, for example, the sum of two such functions is not in the set.

- We prove that

$$\|y\|_1 = \int_a^b |y(x)| dx$$

is a norm on the set of continuous functions on the interval $[a, b]$. First

$$\|\alpha y\|_1 = \int_a^b |\alpha y(x)| dx = |\alpha| \int_a^b |y(x)| dx = |\alpha| \|y\|_1.$$

Next, if $\|y\|_1 = 0$ iff $\int_a^b |y(x)| dx = 0$ iff $y(x) = 0$. Finally, the triangle inequality is proved by

$$\begin{aligned} \|y + v\|_1 &= \int_a^b |y(x) + v(x)| dx \leq \int_a^b (|y(x)| + |v(x)|) dx \\ &= \int_a^b |y(x)| dx + \int_a^b |v(x)| dx = \|y\|_1 + \|v\|_1. \end{aligned}$$

The proof that the maximum norm is, in fact, a norm, follows in the same manner. To prove the triangle inequality use the fact that the maximum of a sum is less than or equal to the sum of the maxima.

- Let $y_1 = 0$ and $y_2 = 0.01 \sin 1000x$. Then

$$\|y_1 - y_2\|_s = \max |0.01 \sin 1000x| = 0.01$$

and

$$\|y_1 - y_2\|_w = \max |0.01 \sin 1000x| + \max |(0.01)(1000) \cos 1000x| = 10.01.$$

- We have

$$\begin{aligned} \delta J(y_0, \alpha h) &= \lim_{\varepsilon \rightarrow 0} \frac{J(y_0 + \varepsilon \alpha h) - J(y_0)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \alpha \frac{J(y_0 + \varepsilon \alpha h) - J(y_0)}{\varepsilon \alpha} \\ &= \lim_{\eta \rightarrow 0} \alpha \frac{J(y_0 + \eta h) - J(y_0)}{\eta} \\ &= \alpha \delta J(y_0, h). \end{aligned}$$

5. (a) not linear; (b) not linear; (c) not linear; (d) not linear; (e) linear; (f) not linear.
6. An alternate characterization of continuity of a functional that is often easier to work with is: a functional J on a normed linear space with norm $\|\cdot\|$ is continuous at y if for any sequence of functions y_n with $\|y_n - y\| \rightarrow 0$ we have $J(y_n) \rightarrow J(y)$ as $n \rightarrow \infty$.
- (a) Now let $\|y_n - y\|_w \rightarrow 0$. Then $\|y_n - y\|_s \rightarrow 0$ (because $\|v\|_s \leq \|v\|_w$). By assumption J is continuous at y in the strong norm, and therefore $J(y_n) \rightarrow J(y)$. So J is continuous in the weak norm.
- (b) Consider the arclength functional $J(y) = \int_a^b \sqrt{1 + (y')^2} dx$. If two functions are close in the strong norm, i.e., if the maximum of their difference is small, then it is not necessarily true that their arclengths are close. For example, one may oscillate rapidly while the other does not. Take $y = 0$ and $y = n^{-1} \sin nx$ for large n . These two functions are close in the strong norm, but not the weak norm.
7. (a) $\delta J(y, h) = \int_a^b (hy' + yh') dx$.
 (b) $\delta J(y, h) = \int_a^b (2h'y' + 2h) dx$.
 (c) $\delta J(y, h) = e^{y(a)} h(a)$.
 (d) See Exercise 3.6.
 (e) $\delta J(y, h) = \int_a^b h(x) \sin x dx$.
 (f) $\delta J(y, h) = \int_a^b 2y'h' dx + G'(y(b))h(b)$.
8. Let $y_n \rightarrow y$. Then $J(y_n) = J(y_n - y) + J(y) \rightarrow J(y)$ because $J(y_n - y) \rightarrow 0$ (since $y_n - y \rightarrow 0$ and J is continuous at zero by assumption).
9. We have

$$J(y + \epsilon h) = \int_a^b (x(y' + \epsilon h')^2 + (y + \epsilon h) \sin(y' + \epsilon h')^2) dx.$$

Now take the second derivative of this function of ϵ with respect to ϵ and then set $\epsilon = 0$. We obtain

$$\delta^2 J = \int_a^b (2x(h')^2 - y(h')^2 \sin y' + 2hh' \cos y') dx.$$

10. Here the functional is

$$J(y) = \int_0^1 (x^2 - y^2 + (y')^2) dx.$$

Then

$$\delta J(y, h) = \int_0^1 (-2yh + 2y'h') dx.$$

Thus

$$\delta J(x, x^2) = \frac{3}{2},$$

and

$$\Delta J = J(y + \varepsilon h) - J(y) = J(x + \varepsilon x^2) - J(x) = \text{etc.}$$

11. $J(y) = \int_0^1 (1+x)y^2 dx$. We find

$$\delta J(y, h) = 2 \int_0^1 (1+x)y'h' dx.$$

Now substitute the given y to show $\delta J(y, h) = 0$ for appropriate h .

12. In this case

$$J(y) = \int_0^{2\pi} (y')^2 dx.$$

Then

$$J(y + \varepsilon h) = \int_0^{2\pi} (1 + \varepsilon \cos x)^2 dx = 2\pi + \varepsilon^2 \int_0^{2\pi} \cos^2 x dx.$$

Then

$$\frac{d}{d\varepsilon} J(y + \varepsilon h) = 2\varepsilon \int_0^{2\pi} \cos^2 x dx,$$

and so

$$\left. \frac{d}{d\varepsilon} J(y + \varepsilon h) \right|_{\varepsilon=0} = 0.$$

Thus, by definition, J is stationary at $y = x$ in the direction $h = \sin x$. The family of curves $y + \varepsilon h$ is shown in the figure.

13. Here

$$J(y) = \int_0^1 (3y^2 + x) dx + y(0)^2.$$

Then

$$\delta J(y, h) = \int_0^1 6yh dx + 2y(0)h(0).$$

Substituting $y = x$ and $h = x + 1$ gives $\delta J = 5$.

3.3 The Simplest Problem

Exercises, page 175

- (a) The Euler equation reduces to an identity ($0 = 0$), and hence every C^2 function is an extremal. (b) The Euler equation reduces to an identity ($0 = 0$), and hence every C^2 function is an extremal. (c) The Euler equation reduces to $y = 0$, which is the only extremal. Remember, by definition, solutions to the Euler equation are extremals, regardless of the boundary conditions.

2. (a) The Euler equation is

$$L_y - \frac{d}{dx}L_{y'} = \frac{d}{dx}(2y'/x^3) = 0$$

Thus

$$y(x) = Ax^4 + B$$

- (b) The Euler equation is

$$y'' - y = e^x$$

The general solution is

$$y(x) = Ae^x + Be^{-x} + \frac{x}{2}e^x$$

3. The Euler equation is

$$L_y - \frac{d}{dx}L_{y'} = f_y\sqrt{1+(y')^2} - \frac{d}{dx}\frac{fy'}{\sqrt{1+(y')^2}} = 0.$$

Taking the total derivative and then multiplying by $\sqrt{1+(y')^2}$ gives

$$f_y - y'f_x - fy'' + \frac{f(y')^2y''}{1+(y')^2} = 0$$

This reduces to

$$f_y - y'f_x - \frac{fy''}{1+(y')^2} = 0$$

4. The Euler equation is

$$-\frac{d}{dx}L_{y'} = 0$$

or

$$\frac{y'}{x\sqrt{1+(y')^2}} = c$$

Solving for y' (take the positive square root since, by the boundary conditions, we want $y' > 0$), separating variables, and integrating yields

$$y(x) = \int \sqrt{\frac{c^2x^2}{1-c^2x^2}} dx + k$$

Then make the substitution $u = 1 - c^2x^2$ to perform the integration. We obtain

$$y(x) = \frac{1}{c}\sqrt{1-c^2x^2} + k$$

Applying the boundary conditions to determine the constants finally leads to

$$y(x) = -\sqrt{5-x^2} + 2$$

which is an arc of a circle.

5. Notice that, by expanding, combining terms, and using the Euler equation,

$$\begin{aligned}
 J(Y+h) - J(Y) &= \int_a^b [2pY'h' + 2qYh + ph'^2 + qh^2] dx \\
 &> \int_a^b [2pY'h' + 2qYh] dx \\
 &= 2pY'h \Big|_a^b - \int_a^b [(2pY')'h - 2qYh] dx \\
 &= 0.
 \end{aligned}$$

The last two lines follow from integration by parts and the fact that Y satisfies the Euler equation

$$(pY')' - qY = 0.$$

6. Let $h \in C^2$. Then

$$\begin{aligned}
 \delta J(y, h) &= \int_a^b \int_a^b K(s, t)[y(s)h(t) + h(s)y(t)] ds dt \\
 &\quad + 2 \int_a^b y(t)h(t) dt - 2 \int_a^b h(t)f(t) dt
 \end{aligned}$$

Now, using the symmetry of K and interchanging the order of integration allows us to rewrite the first integral as

$$2 \int_a^b \int_a^b K(s, t)y(s)h(t) ds dt$$

Then

$$\delta J(y, h) = 2 \int_a^b \left(\int_a^b K(s, t)y(s) ds + y(t) - f(t) \right) h(t) dt$$

Thus

$$\int_a^b K(s, t)y(s) ds + y(t) - f(t) = 0$$

This is a Fredholm integral equation (see Chapter 4) for y .

7. The Euler equation is

$$-((1+x)y')' = 0$$

or

$$y' = c_1/(1+x)$$

Integrating again

$$y(x) = c_1 \ln(1+x) + c_2$$

If $y(0) = 0$, $y(1) = 1$ then we get

$$y(x) = \frac{\ln(1+x)}{\ln 2}$$

If the boundary condition at $x = 1$ is changed to $y'(1) = 0$, then the extremal is $y(x) \equiv 0$.

8. In each case we minimize the functional $T(y)$ given in Example 3.18 on page 174.

- (a) When $n = kx$ the Lagrangian is independent of y , and therefore the Euler equation reduces to

$$x\sqrt{1+y'^2} = C.$$

Separating variables and integrating gives

$$y = \int \sqrt{\frac{C_1 - x^2}{x^2}} dx + C_2.$$

The right hand side can be integrated using a trigonometric substitution.

- (b) The integrand is independent of x .
 (c) The integrand is independent of x .
 (d) This problem is similar to the brachistochrone problem (see Example 3.17).

9. Observe that

$$\begin{aligned} L_t - \frac{d}{dt}(L - y'L_{y'}) &= L_t - \frac{dL}{dt} + y' \frac{dL_{y'}}{dt} + y'' L_{y'} \\ &= L_t - L_t - L_{y'} y' - L_{y''} y'' + y' \frac{dL_{y'}}{dt} + y'' L_{y'} \\ &= -y'(L_{y'} - \frac{d}{dt} L_{y'}) \end{aligned}$$

10. The minimal surface of revolution is found by minimizing

$$J(y) = \int_a^b 2\pi y \sqrt{1 + (y')^2} dx$$

Because the integrand does not depend explicitly on x , a first integral is

$$L - y'L_{y'} = c$$

Upon expanding and simplifying, this equation leads to

$$\frac{dy}{dx} = \sqrt{k^2 y^2 - 1}$$

Now separate variables and integrate while using the fact that

$$\frac{d}{du} \cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}}.$$

11. The Euler equation becomes

$$x^2 y'' + 2xy' - y = 0$$

which is a Cauchy-Euler equation. Its solution is

$$y(x) = Ax^{(-1+\sqrt{5})/2} + Bx^{(-1-\sqrt{5})/2}$$

(recall that a Cauchy-Euler equation can be solved by trying power functions, $y = x^m$ for some m).

12. To find Euler's equation, use the fact that L is independent of x and therefore $L - y' L_{y'} = C$.
13. Using the Euler equation it is straight forward to see that the extremal is $Y = 0$, giving the value $J(Y) = 4$. Take, for example, $y = \sin \pi x$. Then $J(y) > 4$. Hence, $Y = 0$ does not give a local maximum. Because the extremals are only necessary conditions, there is no guarantee that $Y = 0$ provides a minimum either. Note also that, for any y , we have $J(y) = \int_0^1 [y^2 + (y' - 2)^2] dx \geq 0$.

14. Substitution of r into the integral gives the variational problem

$$E(y) = 2 \int_0^T e^{-\beta t} \sqrt{\alpha y - y'} dt \rightarrow \max.$$

The Euler equation is

$$\frac{1 - \beta}{\sqrt{\alpha y - y'}} - \frac{d}{dt} \frac{1}{\sqrt{\alpha y - y'}} = 0.$$

This simplifies to

$$y'' + (\alpha + 2\beta - 2)y' + 2\alpha(1 - \beta)y = 0,$$

which is a linear equation with constant coefficients.

3.4 Generalizations

Exercises, page 184

1. (a) The Euler equations are

$$8y_1 - y_2'' = 0, \quad 2y_2 - y_1'' = 0.$$

Eliminating y_2 gives

$$y_1^{(4)} - 16y_1 = 0.$$

The characteristic equation is $m^4 - 16 = 0$, which has roots $m = \pm 2, \pm 2i$. Therefore the solution is

$$y_1(x) = ae^{2x} + be^{-2x} + c \cos 2x + d \sin 2x,$$

where a, b, c, d are arbitrary constants. Then $y_2 = 0.5y_1''$, and the four constants a, b, c, d can be computed from the boundary conditions.

(b) The Euler equation is

$$y^{(4)} = 0,$$

and thus the extremals are

$$y(x) = a + bx + cx^2 + dx^3.$$

The four constants a, b, c, d can be computed from the boundary conditions.

(c) The Euler equation is $y^{(4)} - 2y'' + y = 0$.

(d) The Euler equation is $y^{(4)} = 0$.

(e) The Euler equation is $y^{(4)} - y^{(3)} - y'' - y = 0$.

2. The Euler equation for $J(y) = \int L(x, y, y', y'')dx$ is

$$L_y - (L_{y'})' + (L_{y''})'' = 0$$

If $L_y = 0$ then clearly

$$L_{y'} - (L_{y''})' = \text{const.}$$

If $L_x = 0$, then expand all the derivatives and use the Euler equation to show

$$\frac{d}{dx} (L - y'(L_{y'} - (L_{y''})' - y''L_{y''})) = 0$$

3. The Euler equation is

$$L_M - \frac{d}{dx} L_{M'} = 2a(aM - M' - b) + \frac{d}{dx} 2(aM - M') = 0,$$

or

$$M'' - a^2M = -ab.$$

The general solution is

$$M(t) = Ae^{at} + Be^{-at} + \frac{b}{a}.$$

The left boundary condition is $M(0) = M_0$; the right boundary condition is the natural boundary condition $L_{M'} = 0$ at $t = T$, or

$$aM(T) - M'(T) = b.$$

4. The two Euler equations (expanded out) are

$$\begin{aligned}L_{y'y''} + L_{y'z'}z'' &= 0 \\L_{z'y''} + L_{z'z'}z'' &= 0\end{aligned}$$

It is given that the determinant of the coefficient matrix of this system is nonzero. Therefore the only solution is $y'' = z'' = 0$, which gives linear functions for y and z .

5. (a) The natural boundary condition is $y'(1) + y(1) = 0$. The extremals are

$$y(x) = ae^x + be^{-x}$$

. The boundary conditions force $a + b = 1$ and $a = 0$, so $y(x) = e^{-x}$.

- (b) The Euler equation is

$$y'' + 2y' + y = 0,$$

giving extremals

$$y(x) = ae^{-x} + bxe^{-x}.$$

The boundary conditions are $y(0) = 1$ and $y'(3) = 0$.

- (c) The Euler equation is

$$x^2y'' + 2xy' + \frac{1}{4}y = 0,$$

which is a Cauchy-Euler equation. Assuming solutions of the form $y = t^m$ gives the characteristic equation

$$m(m-1) + 2m + \frac{1}{4} = 0,$$

which has a real double root $m = -\frac{1}{2}$. Thus the general solution is

$$y(x) = a\frac{1}{\sqrt{x}} + b\frac{1}{\sqrt{x}}\ln x.$$

The given boundary condition is $y(1) = 1$; the natural boundary condition at $x = e$ is $y'(e) = 0$. One finds from these two conditions that $a = b = 1$.

- (d) The Euler equation is

$$y'' - 2y' = -1.$$

- (e) The extremals are

$$y(x) = ax + b.$$

The boundary conditions are $y(0) = 1$ and $y'(1) + y(1) = 0$.

6. The natural boundary condition is

$$L_{y'}(b, y(b), y'(b)) + G'(y(b)) = 0.$$

7. The Lagrangian is $L = \sqrt{1 - k^2 + y'^2} - ky'$, where $0 < k < 1$. It does not depend on x , and therefore a first integral is

$$\frac{y'}{\sqrt{1 - k^2 - y'^2}} + k = C.$$

Solving gives $y(x) = Ax + B$. Now $y(0) = 0$ forces $B = 0$ and $y(x) = Ax$. Now apply the natural boundary condition $L'_y = 0$ at $x = b$. We get

$$\frac{y'}{\sqrt{1 - k^2 - y'^2}} + k = \frac{A}{\sqrt{1 - k^2 - A^2}} + k = 0.$$

This gives

$$A = -\sqrt{\frac{k^2(1 - k^2)}{1 + k^2}}.$$

8. The natural boundary condition is (see Problem 6)

$$9y'(2) + 9y(2) = 2.$$

3.5 The Canonical Formalism

Exercises, page 196

1. The Hamiltonian is

$$H(t, y, p) = \frac{p^2}{4r(t)} - q(t)y^2$$

Hamilton's equations are

$$y' = \frac{p}{2r(t)}, \quad p' = 2q(t)y$$

2. Here we have

$$J(y) = \int \sqrt{(t^2 + y^2)(1 + (y')^2)}$$

We find

$$p = L_{y'} = \frac{\sqrt{t^2 + y^2}y'}{\sqrt{1 + (y')^2}}$$

which yields

$$(y')^2 = \frac{p^2}{t^2 + y^2 - p^2}$$

The Hamiltonian simplifies, after some algebra, to

$$H(t, y, p) = -\sqrt{t^2 + y^2 - p^2}$$

Then Hamilton's equations are

$$\frac{dy}{dt} = \frac{p}{\sqrt{t^2 + y^2 - p^2}}, \quad \frac{dp}{dt} = \frac{y}{\sqrt{t^2 + y^2 - p^2}}$$

Dividing the two equations gives

$$\frac{dy}{dp} = \frac{p}{y}$$

Integrating yields

$$y^2 - p^2 = \text{const}$$

These are hyperbola in the yp phase plane.

3. Hamilton's equations for the pendulum are

$$\theta' = \frac{p}{ml^2}, \quad p' = -mgl \sin \theta$$

4. (a) The potential energy is the negative integral of the force, or

$$V(y) = -\int (-\omega^2 y + ay^2) dy = \frac{1}{2}\omega^2 y^2 - \frac{1}{3}ay^3$$

The Lagrangian is $L = \frac{1}{2}m(y')^2 - V(y)$. The Euler equation coincides with Newton's second law:

$$my'' = -\omega^2 y + ay^2$$

(b) The momentum is $p = L_{y'} = my'$, which gives $y' = p/m$. The Hamiltonian is

$$H(y, p) = \frac{1}{2}(p/m)^2 + V(y)$$

which is the kinetic plus the potential energy, or the total energy of the system. Yes, energy is conserved (L is independent of time).

(c) We have $H = E$ for all time t , so at $t = 0$ we have

$$\frac{1}{2}(p(0)/m)^2 + V(y(0)) = \frac{\omega^2}{10}$$

If $y(0) = 0$ we can solve for $p(0)$ to get the momentum at time zero; but this gives

$$y'(0) = \pm\sqrt{\omega^2/5m}$$

- (d) The potential energy has a local maximum at $y = \omega^2/a$ and is equal to

$$V_{max} = V(\omega^2/a) = \frac{\omega^6}{6a^2}$$

Note that $V = 0$ at $y = 0, 3\omega^2/2a$. If $E < V_{max}$ then we obtain oscillatory motion; if $E > V_{max}$, then the motion is not oscillatory. Observe that the phase diagram (the solution curves in yp -space, or phase space), can be found by graphing

$$p = \pm \sqrt{2m(E - V(y))}$$

for various constants E .

5. The Euler equations are

$$F_y - \frac{d}{dt}F_{y'} = 0, \quad F_p - \frac{d}{dt}F_{p'} = 0,$$

where $F = py' - H(t, y, p)$. Easily we find from these two equations that

$$-H_y - \frac{d}{dt}p = 0, \quad y' - H_p = 0.$$

6. Here the force is $F(t, y) = ke^t/y^2$. We can define a potential by $V(t, y) = -\int F(t, y)dy = ke^t/y$. The Lagrangian is

$$L(t, y, y') = \frac{m}{2}(y')^2 - V(t, y)$$

The Euler equation is

$$my'' - ke^t/y^2 = 0$$

which is Newton's second law of motion. The Hamiltonian is

$$H(t, y, p) = \frac{p^2}{2m} + \frac{k}{y}e^t$$

which is the total energy. Is energy conserved? We can compute dH/dt to find

$$\frac{dH}{dt} = ke^t/y \neq 0$$

So the energy is not constant.

7. The kinetic energy is $T = m(y')^2/2$. The force is mg (with a plus sign since positive distance is measured downward). Thus $V(y) = -mgy$. Then the Lagrangian is $L = m(y')^2/2 + mgy$.

8. We have, for example,

$$L_{x_i} - \frac{d}{dt}L_{x'_i} = \frac{\partial V}{\partial x_i} - \frac{d}{dt}(mx'_i) = 0,$$

or

$$mx''_i = F_i.$$

9. We have $r' = -\alpha$ and so $r(t) = -\alpha t + l$. Then the kinetic energy is

$$T = \frac{m}{2}((r')^2 + r^2(\theta')^2) = \frac{m}{2}(\alpha^2 + (l - \alpha t)^2(\theta')^2),$$

and the potential energy is

$$V = mgh = mg(l - r \cos \theta) = mg(l - (l - \alpha t) \cos \theta).$$

The Lagrangian is $L = T - V$. The Euler equation, or the equation of motion, is

$$g \sin \theta + (l - \alpha t)\theta'' - 2\alpha\theta = 0.$$

One can check that the Hamiltonian is not the same as the total energy; energy is not conserved in this system. In fact, one can verify that

$$\frac{d}{dt}(T + V) = -mg\alpha \cos \theta \neq 0.$$

10. Follow the instructions.

11. The Emden-Fowler equation is

$$y'' + \frac{2}{t}y' + y^5 = 0.$$

Multiply by the integrating factor t^2 to write the equation in the form

$$(t^2y')' + t^2y^5 = 0.$$

Now we can identify this with the Euler equation:

$$L_y = -t^2y^5, \quad L_{y'} = t^2y'.$$

Integrate these two equations to find

$$L = \frac{1}{2}t^2(y')^2 - \frac{1}{6}t^2y^6 + \phi(t)$$

12. Multiply $y'' + ay' + b = 0$ by the integrating factor $\exp(at)$ to get

$$(e^{at}y')' + be^{at} = 0$$

Now identify the terms in this equation with the terms in the Euler equation as in Exercise 5.14. Finally we arrive at a Lagrangian

$$L = e^{at} \left(\frac{m}{2}(y')^2 - by \right)$$

There are many Lagrangians, and we have chosen just one by selecting the arbitrary functions.

13. Follow Example 3.27 in the book with m, a, k and y replaced by L, R, C^{-1} and I , respectively.
14. Multiplying the given equation by $\exp P(t)$ makes the sum of the first two terms a total derivative and we get

$$\frac{d}{dt}(e^{P(t)}y') + e^{P(t)}f(y) = 0.$$

Comparing to the Euler equation we take

$$L_y = e^{P(t)}f(y), \quad L_{y'} = -e^{P(t)}y'.$$

Integrating the first gives

$$L = e^{P(t)}F(y) + \phi(t, y'),$$

where ϕ is arbitrary. Plugging this into the second equation we get

$$\phi_{y'} = -e^{P(t)}y'.$$

Integrating,

$$\phi = -\frac{1}{2}y'^2 e^{P(t)} + \psi(t),$$

where ψ is arbitrary. Thus,

$$L = e^{P(t)} \left(F(y) - \frac{1}{2}y'^2 \right) + \psi(t).$$

3.6 Isoperimetric Problems

Exercises, page 203

1. Form the Lagrangian

$$L^* = (y')^2 + \lambda y^2.$$

The Euler equation reduces to

$$y'' - \lambda y = 0, \quad y(0) = y(\pi) = 0$$

The extremals are given by

$$y_n(x) = \pm\sqrt{2/\pi} \sin nx, \quad n = 1, 2, 3, \dots$$

2. Let $L^* = x^2 + (y')^2 + \lambda y^2$. Then the Euler equation for L^* is

$$y'' + \lambda y = 0, \quad y(0) = y(1) = 0.$$

If $\lambda \geq 0$ then this boundary value problem has only trivial solutions. If $\lambda < 0$, say $\lambda = -\beta^2$, then the problem has nontrivial solutions

$$y_n(x) = B_n \sin n\pi x, \quad n = 1, 2, \dots,$$

where the B_n are constants. Applying the constraint gives

$$\int_0^1 B_n^2 \sin^2 n\pi x \, dx = 2.$$

Thus $B_n = \pm 2$ for all n . So the extremals are

$$y_n(x) = \pm 2 \sin n\pi x, \quad n = 1, 2, \dots$$

3. The Euler equations become

$$L_{y_1}^* - \frac{d}{dx} L_{y_1}^* = 0, \quad L_{y_2}^* - \frac{d}{dx} L_{y_2}^* = 0,$$

where $L^* = L + \lambda G$.

4. Form the Lagrangian

$$L^* = xy' - yx' + \lambda \sqrt{(x')^2 + (y')^2}.$$

Because the Lagrangian does not depend explicitly on t , a first integral is given by

$$L^* - x' L_{x'}^* - y' L_{y'}^* = C.$$

5. The problem is to minimize

$$J(y) = \int_0^1 \sqrt{1 + (y')^2} \, dx, \quad y(0) = y(1) = 0$$

subject to the constraint

$$\int_0^1 y(x) \, dx = A.$$

Form the Lagrangian

$$L^* = \sqrt{1 + (y')^2} + \lambda y$$

Because the Lagrangian does not depend explicitly on x , a first integral is given by

$$L^* - y' L_{y'}^* = c.$$

Expanding out this equation leads to

$$y' = \sqrt{\frac{1 - (\lambda y - c)^2}{(\lambda y - c)^2}}.$$

Separating variables and integrating, and then using the substitution $u = 1 - (\lambda y - c)^2$, gives

$$-\frac{1}{2\lambda} \int \frac{du}{\sqrt{u}} = x + c_1.$$

Thus

$$(x + c_1)^2 + (y - c/\lambda)^2 = \frac{1}{\lambda^2},$$

which is a circle. Now the two boundary conditions and the constraint give three equations for the constants c, c_1, λ .

6. Form the Lagrangian

$$L^* = p(y')^2 + qy^2 + \lambda ry^2.$$

Then the Euler equation corresponding to L^* is

$$L_y - (L_{y'})' = 2qy + 2r\lambda y - 2(py')' = 0,$$

or

$$(py')' - qy = r\lambda y, \quad y(a) = y(b) = 0$$

This is a Sturm-Liouville problem for y (see Chapter 4).

7. Solve the constraint equation to obtain

$$z = g(t, y).$$

Substitute this into the functional to obtain

$$W(y) \equiv \int_a^b F(t, y, y') \equiv \int_a^b L(t, y, g(t, y), y', g_t + g_y y') dt.$$

Now form the Euler equation for F . We get

$$F_y - \frac{d}{dt} F_{y'} = 0,$$

or, in terms of L ,

$$L_y + L_z g_y + L_{z'}(g_{ty} + g_{yy} y') - \frac{d}{dt}(L_{y'} + L_{z'} g_y) = 0.$$

This simplifies to

$$L_y - \frac{d}{dt} L_{y'} + g_y(L_z - \frac{d}{dt} L_{z'}) = 0.$$

Also $G(t, y, g(t, y)) = 0$, and taking the partial with respect to y gives

$$G_y + G_z g_y = 0.$$

Thus

$$\frac{L_y - \frac{d}{dt} L_{y'}}{G_y} = \frac{L_z - \frac{d}{dt} L_{z'}}{G_y}.$$

Now these two expressions must be equal to the same function of t , that is

$$L_y - \frac{d}{dt} L_{y'} = \lambda(t) G_y, \quad L_z - \frac{d}{dt} L_{z'} = \lambda(t) G_z.$$