Goldstein Classical Mechanics Notes

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1 Chapter 1: Elementary Principles

1.1 Mechanics of a Single Particle

Classical mechanics incorporates special relativity. 'Classical' refers to the contradistinction to 'quantum' mechanics.

Velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}.$$

Linear momentum:

$$\mathbf{p} = m\mathbf{v}.$$

Force:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

In most cases, mass is constant and force is simplified:

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}.$$

Acceleration:

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}.$$

Newton's second law of motion holds in a reference frame that is inertial or Galilean.

Angular Momentum:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

Torque:

$$\mathbf{T} = \mathbf{r} \times \mathbf{F}.$$

Torque is the time derivative of angular momentum:

$$\mathbf{T} = \frac{d\mathbf{L}}{dt}.$$

Work:

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r}.$$

In most cases, mass is constant and work simplifies to:

$$W_{12} = m \int_1^2 \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt = m \int_1^2 \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} dt = m \int_1^2 \mathbf{v} \cdot d\mathbf{v}$$

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2) = T_2 - T_1$$

Kinetic Energy:

$$T = \frac{mv^2}{2}$$

The work is the change in kinetic energy.

A force is considered conservative if the work is the same for any physically possible path. Independence of W_{12} on the particular path implies that the work done around a closed ciruit is zero:

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0$$

If friction is present, a system is non-conservative.

Potential Energy:

$$\mathbf{F} = -\nabla V(\mathbf{r}).$$

The capacity to do work that a body or system has by viture of is position is called its potential energy. V above is the potential energy. To express work in a way that is independent of the path taken, a change in a quantity that depends on only the end points is needed. This quantity is potential energy. Work is now $V_1 - V_2$. The change is -V.

Energy Conservation Theorem for a Particle: If forces acting on a particle are conservative, then the total energy of the particle, T + V, is conserved.

The Conservation Theorem for the Linear Momentum of a Particle states that linear momentum, **p**, is conserved if the total force **F**, is zero.

The Conservation Theorem for the Angular Momentum of a Particle states that angular momentum, **L**, is conserved if the total torque **T**, is zero.

1.2 Mechanics of Many Particles

Newton's third law of motion, equal and opposite forces, does not hold for all forces. It is called the weak law of action and reaction.

Center of mass:

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M}.$$

Center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Internal forces that obey Newton's third law, have no effect on the motion of the center of mass.

$$\mathbf{F}^{(e)} \equiv M \frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)}.$$

Motion of center of mass is unaffected. This is how rockets work in space.

Total linear momentum:

$$\mathbf{P} = \sum_i m_i \frac{d\mathbf{r}_i}{dt} = M \frac{d\mathbf{R}}{dt}.$$

Conservation Theorem for the Linear Momentum of a System of Particles: If the total external force is zero, the total linear momentum is conserved.

The strong law of action and reaction is the condition that the internal forces between two particles, in addition to being equal and opposite, also lie along the line joining the particles. Then the time derivative of angular momentum is the total external torque:

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}.$$

Torque is also called the moment of the external force about the given point.

Conservation Theorem for Total Angular Momentum: $\mathbf{L}$ is constant in time if the applied torque is zero.

Linear Momentum Conservation requires weak law of action and reaction.

Angular Momentum Conservation requires strong law of action and reaction.

Total Angular Momentum:

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}'_i \times \mathbf{p}'_i.$$

Total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum of motion about the center of mass. If the center of mass is at rest wrt the origin then the angular momentum is independent of the point of reference.

Total Work:

$$W_{12} = T_2 - T_1$$

where T is the total kinetic energy of the system: $T = \frac{1}{2}\sum_i m_i v_i^2$.
Total kinetic energy:

$$T = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_i m_i v_i'^2.$$

Kinetic energy, like angular momentum, has two parts: the K.E. obtained if all the mass were concentrated at the center of mass, plus the K.E. of motion about the center of mass.

Total potential energy:

$$V = \sum_i V_i + \frac{1}{2}\sum_{i,j\ i\neq j} V_{ij}.$$

If the external and internal forces are both derivable from potentials it is possible to define a total potential energy such that the total energy $T + V$ is conserved.

The term on the right is called the internal potential energy. For rigid bodies the internal potential energy will be constant. For a rigid body the internal forces do no work and the internal potential energy remains constant.

1.3 Constraints

- *holonomic constraints*: think rigid body, think $f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, ..., t) = 0$, think a particle constrained to move along any curve or on a given surface.
- *nonholonomic constraints*: think walls of a gas container, think particle placed on surface of a sphere because it will eventually slide down part of the way but will fall off, not moving along the curve of the sphere.

1. *rheonomous constraints*: time is an explicit variable...example: bead on moving wire
2. *scleronomous constraints*: equations of contraint are NOT explicitly dependent on time...example: bead on rigid curved wire fixed in space

Difficulties with constraints:

1. Equations of motion are not all independent, because coordinates are no longer all independent

2. Forces are not known beforehand, and must be obtained from solution.

For *holonomic constraints* introduce **generalized coordinates**. Degrees of freedom are reduced. Use independent variables, eliminate dependent coordinates. This is called a *transformation*, going from one set of dependent variables to another set of independent variables. Generalized coordinates are worthwhile in problems even without constraints.

Examples of generalized coordinates:

1. Two angles expressing position on the sphere that a particle is constrained to move on.

2. Two angles for a double pendulum moving in a plane.

3. Amplitudes in a Fourier expansion of $\mathbf{r}_j$.

4. Quanities with with dimensions of energy or angular momentum.

For *nonholonomic constraints* equations expressing the constraint cannot be used to eliminate the dependent coordinates. Nonholonomic constraints are HARDER TO SOLVE.

1.4 D'Alembert's Principle and Lagrange's Equations

Developed by D'Alembert, and thought of first by Bernoulli, the principle that:

$$\sum_i (\mathbf{F}_i^{(a)} - \frac{d\mathbf{p}_i}{dt}) \cdot \delta \mathbf{r}_i = 0$$

This is valid for systems which virtual work of the forces of constraint vanishes, like rigid body systems, and no friction systems. This is the only restriction on the nature of the constraints: workless in a virtual displacement. This is again D'Alembert's principle for the motion of a system, and what is good about it is that the forces of constraint are not there. This is great news, but it is not yet in a form that is useful for deriving equations of motion. Transform this equation into an expression involving virtual displacements of the generalized coordinates. The generalized coordinates are independent of each other for holonomic constraints. Once we have the expression in terms of generalized coordinates the coefficients of the δq_i can be set separately equal to zero. The result is:

$$\sum \{ [\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j}] - Q_j \} \delta q_j = 0$$

Lagrange's Equations come from this principle. If you remember the individual coefficients vanish, and allow the forces derivable from a scaler potential function, and forgive me for skipping some steps, the result is:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = 0$$

1.5 Velocity-Dependent Potentials and The Dissipation Function

The velocity dependent potential is important for the electromagnetic forces on moving charges, the electromagnetic field.

$$L = T - U$$

where U is the generalized potential or velocity-dependent potential.

For a charge mvoing in an electric and magnetic field, the Lorentz force dictates:

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})].$$

The equation of motion can be dervied for the x-dirction, and notice they are identical component wise:

$$m\ddot{x} = q[E_x + (\mathbf{v} \times \mathbf{B})_x].$$

If frictional forces are present(not all the forces acting on the system are derivable from a potential), Lagrange's equations can always be written:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = Q_j.$$

where Q_j represents the forces not arising from a potential, and L contains the potential of the conservative forces as before.

Friction is commonly,

$$F_{fx} = -k_x v_x.$$

Rayleigh's dissipation function:

$$F_{dis} = \frac{1}{2}\sum_i (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2).$$

The total frictional force is:

$$\mathbf{F}_f = -\nabla_v F_{dis}$$

Work done by system against friction:

$$dW_f = -2F_{dis}dt$$

The rate of energy dissipation due to friction is $2F_{dis}$ and the component of the generalized force resulting from the force of friction is:

$$Q_j = -\frac{\partial F_{dis}}{\partial \dot{q}_j}.$$

In use, both L and F_{dis} must be specified to obtain the equations of motion:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = -\frac{\partial F_{dis}}{\partial \dot{q}_j}.$$

1.6 Applications of the Lagrangian Formulation

The Lagrangian method allows us to eliminate the forces of constraint from the equations of motion. Scalar functions T and V are much easier to deal with instead of vector forces and accelerations.

Procedure:

1. Write T and V in generalized coordinates.
2. Form L from them.
3. Put L into Lagrange's Equations
4. Solve for the equations of motion.

Simple examples are:

1. a single particle is space(Cartesian coordinates, Plane polar coordinates)
2. atwood's machine
3. a bead sliding on a rotating wire(time-dependent constraint).

Forces of contstraint, do not appear in the Lagrangian formulation. They also cannot be directly derived.