Principles of Mathematical Physics

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These are the notes accompanying the first half of the lectures of **Principles of Mathematical Physics**. These notes are still in a state of flux and I am happy to receive comments and suggestions either by email or in person.

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List of lectures

Lecture 1: Newtonian mechanics

I think that Isaac Newton is doing most of the driving right now.
— Bill Anders, Apollo 8 mission

In this lecture we introduce the basic assumptions underlying newtonian mechanics, which deals with the motions of point particles in space. Being based on empirical evidence, these assumptions have a limited domain of validity and hence the laws derived from them are known to break down in the very small, the very large or the very fast. Nevertheless newtonian mechanics has a remarkably wide domain of applicability, encompassing for instance both apples falling on the surface of the Earth and planets orbiting stars. Historically it was also the first modern physical theory.

1.1 The universe according to Newton

The newtonian universe is $\mathbb{R}\times\mathbb{R}^3$, where $\mathbb{R}$ is **time** and

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{such that} \quad x_i \in \mathbb{R} \right\} \tag{1}$$

is a three-dimensional euclidean space *together* with the usual dot product,

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i \ , \tag{2}$$

for $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. The dot product defines a **norm** on $\mathbb{R}^3$. If $\boldsymbol{a}\in\mathbb{R}^3$, its norm $|\boldsymbol{a}|$ is defined by

$$|\boldsymbol{a}| = (\boldsymbol{a}\cdot\boldsymbol{a})^{1/2} \ . \tag{3}$$

> **Notation**
>
> This seems a good point to alert you to other notations you will come across. Our convention is that vectors are **boldfaced** in the printed notes, but underlined in the blackboard. In Physics, it is also common to write a_i for (the components of) the vector $\boldsymbol{a}$, and the scalar product $\boldsymbol{a}\cdot\boldsymbol{b}$ is then written $a_i b_i$ with the convention that repeated indices are to be summed over all their values, in this case $i = 1,2,3$. Finally, an alternative notation for the scalar product is $\langle \boldsymbol{a}, \boldsymbol{b}\rangle$ which is closer to the bracket notation employed in Quantum Mechanics.

A point $(t,\boldsymbol{a})$ in the universe is called an **event**. Two events $(t,\boldsymbol{a})$ and $(t',\boldsymbol{b})$ are said to be **simultaneous** if and only if $t = t'$. It makes sense to talk about the **distance** between simultaneous events $(t,\boldsymbol{a})$ and $(t,\boldsymbol{b})$, and this is given by the norm $|\boldsymbol{a}-\boldsymbol{b}|$ of the difference vector.

Particle trajectories are given by **worldlines**, which are graphs of functions $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^3$; that is, subsets of the universe of the form

$$\{(t, \boldsymbol{x}(t)) \mid t \in \mathbb{R}\} \ . \tag{4}$$

We will assume that such functions $\boldsymbol{x}$ are continuously differentiable as many times as required. Figure 1 illustrates the worldlines of two particles.

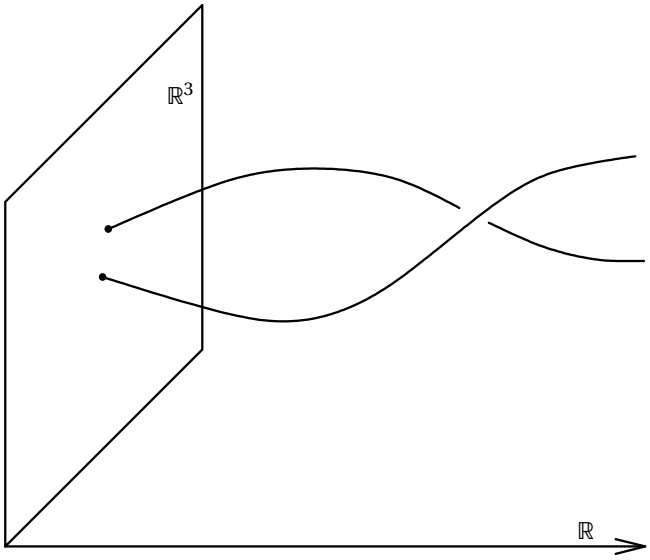


Figure 1: Two worldlines

Let $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^3$ define the worldline of a particle. The first derivative (with respect to time) $\dot{\boldsymbol{x}}$ is called the **velocity** and the second derivative $\ddot{\boldsymbol{x}}$ the **acceleration**.
We are often interested in mechanical systems consisting of more that one particle. The **configuration space** of an n-particle system is the n-fold cartesian product

$$\underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{n} = \mathbb{R}^{\mathrm{N}} \ , \qquad \mathrm{N} = 3n \ .$$

The worldline of the ith particle is given by $\boldsymbol{x}_i : \mathbb{R} \to \mathbb{R}^3$ and the n worldlines together define a curve $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^{\mathrm{N}}$ in the configuration space,

$$\boldsymbol{x}(t) = (\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)) \ .$$

1.2 Newton's equation

The other basic assumption of newtonian mechanics is **determinacy**, which means that the **initial state** of a mechanical system, by which we mean the totality of the positions and velocities of all the particles at a given instant in time, uniquely determines the motion. In other words, $\boldsymbol{x}(0)$ and $\dot{\boldsymbol{x}}(0)$ determine $\boldsymbol{x}(t)$ for all t, or at least for all t in some finite interval.
In particular, the acceleration is determined, so there must exist some relationship of the form

$$\boxed{\ddot{\boldsymbol{x}} = \boldsymbol{\Phi}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)} \ , \tag{5}$$

for some function $\mathbf{\Phi} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$. This second-order ordinary differential equation (ODE) is called **Newton's equation**. Solving a second-order ODE involves integrating twice, which gives rise to two constants of integration (per degree of freedom). These constants are then fixed by the initial conditions.
We will be dealing almost exclusively with functions $\mathbf{\Phi}$ depending only on $\boldsymbol{x}$ and neither on $\dot{\boldsymbol{x}}$ nor on t.

Example 1.1 (Particle in a force field)**.** The version of Newton's equation (5) which describes the motion of a particle in the presence of a force field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is

$$\mathbf{F}(\boldsymbol{x}) = m\ddot{\boldsymbol{x}} \tag{6}$$

where m is the **(inertial) mass** of the particle. A point $\boldsymbol{x}_0 \in \mathbb{R}^3$ where $\mathbf{F}(\boldsymbol{x}_0) = \mathbf{0}$ is said to be a point of **equilibrium**, since a particle sitting at $\boldsymbol{x}_0$ feels no force.

Dimensional analysis

Physical quantities have dimension. The basic dimensions in these lectures are **length** (L), **time** (T) and **mass** (M). For example, the position vector $\boldsymbol{x}$ has dimension of length, and we write this as $[\boldsymbol{x}] = \mathrm{L}$. Similarly, the time-derivative has dimension of inverse time, whence if a time-dependent quantity Q has dimension [Q], then its time-derivative has dimension $[\dot{\mathrm{Q}}] = [\mathrm{Q}]\mathrm{T}^{-1}$. It follows from this that the velocity and acceleration have dimensions $[\dot{\boldsymbol{x}}] = \mathrm{LT}^{-1}$ and $[\ddot{\boldsymbol{x}}] = \mathrm{LT}^{-2}$, respectively. Dimension is multiplicative in the sense that $[\mathrm{Q}_1\mathrm{Q}_2] = [\mathrm{Q}_1][\mathrm{Q}_2]$, whence from Newton's equation (6) $[\mathbf{F}] = [m\ddot{\boldsymbol{x}}] = \mathrm{MLT}^{-2}$, where we have used that $[m] = \mathrm{M}$, naturally. It is a very useful check of the correctness of a calculation that the result should have the expected dimension.

Example 1.2 (Invariance under time reversal)**.** When $\mathbf{\Phi}$ only depends on $\boldsymbol{x}$, Newton's equation (5) is invariant under time reversal; that is, if $\boldsymbol{x}(t)$ solves the equation, so does $\overline{\boldsymbol{x}}(t) := \boldsymbol{x}(-t)$. To see this, it suffices to observe that the double derivative with respect to t is the same as the double derivative with respect to $-t$. In detail,

$$\ddot{\overline{\boldsymbol{x}}}(t) = \ddot{\boldsymbol{x}}(-t) = \mathbf{\Phi}(\boldsymbol{x}(-t)) = \mathbf{\Phi}(\overline{\boldsymbol{x}}(t)) \,,$$

where the second equality follows because $\boldsymbol{x}$ satisfies Newton's equation.

Example 1.3 (The free particle)**.** This is a particular case of Example 1.1, where $\mathbf{F} = \mathbf{0}$. Newton's equation (6) says that there is no acceleration, so that the velocity $\boldsymbol{v}$ is constant. Integrating a second time we obtain

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + t\boldsymbol{v} \,, \tag{7}$$

where $\boldsymbol{x}_0 = \boldsymbol{x}(0)$ is the initial position. Given $\boldsymbol{x}_0$ and $\boldsymbol{v}$ there is a unique solution $\boldsymbol{x}(t)$ to Newton's equation with $\mathbf{F} = \mathbf{0}$ with $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\dot{\boldsymbol{x}}(0) = \boldsymbol{v}$.

Example 1.4 (Circular motion)**.** A particle of mass m is observed moving in a circular trajectory

$$\boldsymbol{x}(t) = \begin{pmatrix} \mathrm{R}\cos\omega t \\ \mathrm{R}\sin\omega t \\ 0 \end{pmatrix}, \tag{8}$$

where R, ω are positive constants. The force is given by $\mathbf{F} = m\ddot{\boldsymbol{x}}$, whence

$$\mathbf{F} = m\ddot{\boldsymbol{x}} = -m\omega^2\boldsymbol{x}\,.$$

Thus the force is parallel to the line joining the origin with $\boldsymbol{x}$ and pointing towards the origin.

A standard trick allows us to turn Newton's equation (5) into an equivalent first-order ODE. The trick consists in introducing a new function $\boldsymbol{v} : \mathbb{R} \to \mathbb{R}^{\mathrm{N}}$ together with the equation $\dot{\boldsymbol{x}} = \boldsymbol{v}$. Newton's equation is then

$$\ddot{\boldsymbol{x}} = \dot{\boldsymbol{v}} = \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{v}, t)\,.$$

In other words, in terms of the function $(\boldsymbol{x}, \boldsymbol{v}) : \mathbb{R} \to \mathbb{R}^{2\mathrm{N}}$, Newton's equation becomes

$$\boxed{(\dot{\boldsymbol{x}}, \dot{\boldsymbol{v}}) = (\boldsymbol{v}, \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{v}, t))\,.} \tag{9}$$

It is not difficult to show that equations (5) and (9) have exactly the same solutions. Indeed, if $\boldsymbol{x}$ solves equation (5), let $\boldsymbol{v} = \dot{\boldsymbol{x}}$ and then $(\boldsymbol{x}, \boldsymbol{v})$ solves (9). Conversely, if $(\boldsymbol{x}, \boldsymbol{v})$ solves (9), then $\boldsymbol{v} = \dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}} = \dot{\boldsymbol{v}} = \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{v}, t) = \boldsymbol{\Phi}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$.
The space of positions and velocities, here $\mathbb{R}^{2\mathrm{N}}$, defines the **state space** of the mechanical system. The pair of functions $(\boldsymbol{x}, \boldsymbol{v})$ defines a curve in the space of states, which, if it obeys (9), is called a **physical trajectory**. This reformulation of Newton's equation makes contact with `MAT-2-MAM`, where it is proved that if $\boldsymbol{\Phi}$ is sufficiently differentiable, equation (9) has a unique solution for specified initial conditions $(\boldsymbol{x}(0), \boldsymbol{v}(0) = \dot{\boldsymbol{x}}(0))$, at least in some time interval. In other words, through every point in state space there passes a unique physical trajectory.
This mathematical fact turns out to have an important physical consequence. Let $\boldsymbol{x}(t)$ be a physical trajectory for a particle in a force field for which $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is a point of equilibrium; that is, a point where the force field vanishes. Then if $\dot{\boldsymbol{x}}(0) = \mathbf{0}$ then $\boldsymbol{x}(t) = \boldsymbol{x}_0$ for all t. Indeed, the constant trajectory $\boldsymbol{x}(t) = \boldsymbol{x}_0$ for all t obeys Newton's equation $\ddot{\boldsymbol{x}}(t) = \mathbf{F}(\boldsymbol{x}(t))$ for all t, and satisfies the initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\dot{\boldsymbol{x}}(0) = \mathbf{0}$. By uniqueness of the solutions to this initial value problems, this is the only solution with those initial conditions.

Example 1.5 (Galilean gravity)**.** Consider dropping an apple of mass m from the Tower of Pisa. Empirical evidence suggests that the force of gravity points downwards, is constant and proportional to the mass. Letting $z(t)$ denote the height at time t, Newton's equation is then

$$m\ddot{z} = -mg\,, \tag{10}$$

where g is a constant with dimension $[g] = \mathrm{LT}^{-2}$, and $g \approx 9.8 ms^{-2}$ on the surface of the Earth. We can solve equation (10) by integrating twice

$$z(t) = z_0 + v_0 t - \tfrac{1}{2} g t^2\,.$$

The relevant space of states is the right half-plane

$$\{(z, v) \mid z \geq 0\} \subset \mathbb{R}^2 \,, \tag{11}$$

and the physical trajectories are the parabolas given by

$$(z(t), v(t)) = \left(z_0 + v_0\, t - \tfrac{1}{2} g\, t^2, v_0 - g\, t\right) \,. \tag{12}$$

Some of these trajectories are plotted in Figure 2. Notice that whatever the initial conditions (z_0, v_0) the apple always ends up on the floor. This is contrary to observation (e.g., rockets can break free of Earth's gravity) and indeed it is known that as the distance from the Earth increases, her gravitational pull weakens. This will be corrected in Newton's theory of gravity.

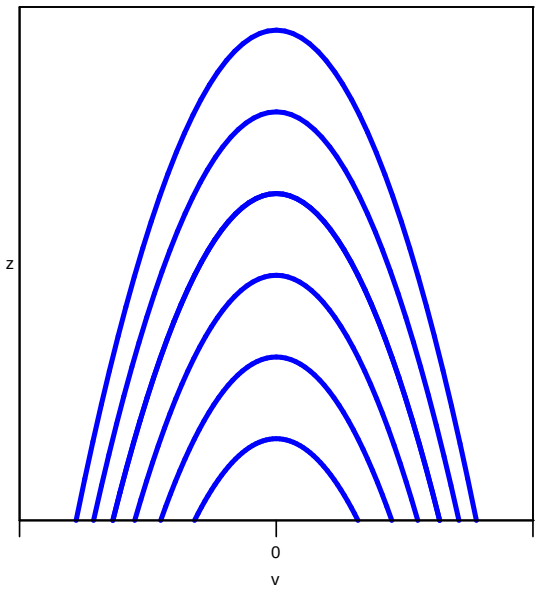


Figure 2: Physical trajectories of equation (10) in units where $g = 1$

The equivalence principle

The m in the RHS of equation (10) is called the **(gravitational) mass** and it is an empirical fact (famously demonstrated by Galileo and later by Eötvös) that it is equal to the (inertial) mass appearing in the LHS. This equality is called the **equivalence principle**: it hints at a geometric origin of gravity and is a cornerstone of Einstein's general theory of relativity.

Lecture 2: Conservation laws

Nature uses as little as possible of anything.
—Johannes Kepler

As a mechanical system evolves in time it will change its state $(\boldsymbol{x}, \boldsymbol{v})$ according to Newton's equation (9). However there are functions of $(\boldsymbol{x}, \boldsymbol{v})$ which remain constant. Such functions are called **integrals** of the motion. Among them there are some which are of particular importance in mechanics. At a fundamental level they are related to symmetries of the physical system: homogeneity of space and time (the fact that there is no preferred origin or initial time) and isotropy of space (the fact that there is no preferred direction), for example. Such an integral of the motion is called a **conserved quantity** due to the fact that it is additive in the sense that, if a mechanical system is composed of two non-interacting parts, then its value for the system is the sum of its values for each of the parts.

Example 2.1 (Conservation of momentum)**.** For the free particle of Example 1.3, the **momentum** $\boldsymbol{p} : \mathbb{R}^6 \to \mathbb{R}^3$, defined by $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{v}) = m\boldsymbol{v}$ is conserved. Of course, $\boldsymbol{v}$ is also conserved, but it is the momentum which is additive. Indeed, if we now consider a system of two non-interacting free particles, with momenta $\boldsymbol{p}_1 = m_1\boldsymbol{v}_1$ and $\boldsymbol{p}_2 = m_2\boldsymbol{v}_2$, the momentum of the system will be the sum $\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2$.

Example 2.2 (Conservation of energy)**.** For the falling apple in Example 1.5 the **energy**

$$\mathrm{E}(z, v) = \tfrac{1}{2}mv^2 + mgz$$

is conserved. Since $z \geq 0$, the energy is non-negative. In this case, the physical trajectories are the parabolas $\frac{1}{2}v^2 + gz = \mathrm{E}/m$, just as we had found by integrating Newton's equation.

2.1 Conservative forces: potentials

In the notation of Example 1.1, a force field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be **conservative** if it can be expressed as (minus) the gradient of a function $\mathrm{V} : \mathbb{R}^3 \to \mathbb{R}$, called the **potential**: $\mathbf{F} = -\nabla \mathrm{V}$. The potential is only defined up to a constant and the minus sign is conventional.

Example 2.3 (Gravitational potential)**.** The gravitational potential in galilean gravity is given by $\mathrm{V} = mgz$. Indeed, computing (minus) the gradient of $\mathrm{V} = mgz$, one finds

$$-\nabla \mathrm{V} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix},$$

as expected.

More generally, for an n-particle mechanical system with configuration space $\mathbb{R}^{\mathrm{N}}$ and state space $\mathbb{R}^{2\mathrm{N}}$, a potential is a function $\mathrm{U} : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ such that Newton's equation (5) can be written as

$$\ddot{\boldsymbol{x}} = -\nabla \mathrm{U}\,.$$

A common feature of conservative force fields is that energy is conserved along physical trajectories. Indeed, Newton's equation (6) for a conservative force field is (after bringing the force term to the LHS):

$$m\ddot{\boldsymbol{x}} + \nabla \mathrm{V} = \mathbf{0} \,.$$

Taking the inner product with $\dot{\boldsymbol{x}}$ we obtain

$$m\dot{\boldsymbol{x}} \cdot \ddot{\boldsymbol{x}} + \dot{\boldsymbol{x}} \cdot \nabla \mathrm{V} = 0 \,,$$

which we recognise as the constancy along physical trajectories of the **energy**

$$\mathrm{E}(\boldsymbol{x}, \boldsymbol{v}) = \tfrac{1}{2} m |\boldsymbol{v}|^2 + \mathrm{V} \,. \tag{13}$$

Indeed, a version of the product (Leibniz) rule says that

$$\frac{d}{dt} |\boldsymbol{v}|^2 = 2\boldsymbol{v} \cdot \dot{\boldsymbol{v}} \,,$$

and the chain rule (see below) says that

$$\frac{d}{dt} \mathrm{V} = \dot{\boldsymbol{x}} \cdot \nabla \mathrm{V} \,,$$

whence along physical trajectories, where $\boldsymbol{v} = \dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}} = -\nabla \mathrm{V}$, we find

$$\frac{d}{dt} \mathrm{E}(\boldsymbol{x}, \boldsymbol{v}) = m\dot{\boldsymbol{x}} \cdot \ddot{\boldsymbol{x}} + \dot{\boldsymbol{x}} \cdot \nabla \mathrm{V} = \dot{\boldsymbol{x}} \cdot (m\ddot{\boldsymbol{x}} + \nabla \mathrm{V}) = 0 \,.$$

The chain rule

Let $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^3$ and $\mathrm{V} : \mathbb{R}^3 \to \mathbb{R}$. The composition $\mathrm{V} \circ \boldsymbol{x} : \mathbb{R} \to \mathbb{R}$ sends $t \in \mathbb{R}$ to $\mathrm{V}(\boldsymbol{x}(t)) \in \mathbb{R}$. The derivative with respect to t of the composition is given by the chain rule, as you have seen in `MAT-2-SVC`:

$$\frac{d}{dt} \mathrm{V}(\boldsymbol{x}(t)) = \nabla \mathrm{V}\big|_{\boldsymbol{x}(t)} \cdot \dot{\boldsymbol{x}}(t) \,. \tag{14}$$

The first term in the RHS of the expression (13) for the energy is called the **kinetic energy** and depends on the motion of the particle, whereas the second term is the **potential energy** and depends on the position. Physical trajectories lie on "constant energy surfaces" in the space of states, defined by

$$\{(\boldsymbol{x}, \boldsymbol{v}) \mid \mathrm{E}(\boldsymbol{x}, \boldsymbol{v}) = \mathrm{E}_0\} \,.$$

In the case of one-dimensional motion energy conservation alone suffices to determine the physical trajectories, as we saw already in the case of galilean gravity.

2.2 Quadratic potentials and simple harmonic motion

We start by considering one-dimensional potential motion. Let $x(t)$ denote the position of a particle of mass m at time t and let $\mathrm{V}(x)$ denote the potential. Newton's equation is then simply

$$m\ddot{x} = -\frac{d\mathrm{V}}{dx}\,. \tag{15}$$

We saw in Example 2.3 that a linear potential is responsible for Galilean gravity, where the force is constant. The next simplest potential is a quadratic potential, which means that the force is linear.

Example 2.4 (Hooke's law)**.** If the potential is $\mathrm{V}(x) = \frac{1}{2}kx^2$, with $k > 0$, the resulting force is $\mathrm{F} = -kx$, which is a good approximation to the restoring force of a spring, an empirical law due to Hooke. In Lecture 7 we will reinterpret this, not as a particular property of springs, but as a universal property of small displacements about stable equilibria.

Hooke's law leads to simple harmonic motion. Indeed, Newton's equation in this case reads

$$m\ddot{x} = -kx\,, \tag{16}$$

whose solutions are

$$x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t\,,$$

where $\omega^2 = k/m$ and $x_0 = x(0)$ and $v_0 = \dot{x}(0)$ are the initial position and velocity of the particle, respectively. Using the trigonometric addition formula

$$\mathrm{A}\sin(\omega t+\varphi) = \mathrm{A}\sin(\omega t)\cos\varphi + \mathrm{A}\cos(\omega t)\sin\varphi$$

and comparing with the solution, we see that

$$x(t) = \mathrm{A}\sin(\omega t+\varphi)$$

where

$$x_0 = \mathrm{A}\sin\varphi \qquad \text{and} \qquad \frac{v_0}{\omega} = \mathrm{A}\cos\varphi\,,$$

whence

$$\mathrm{A}^2 = x_0^2 + \frac{v_0^2}{\omega^2} = \frac{2}{m\omega^2}\mathrm{E} = \frac{2\mathrm{E}}{k}\,,$$

where E is the energy, and

$$\tan\varphi = \frac{\mathrm{A}\sin\varphi}{\mathrm{A}\cos\varphi} = \frac{\omega x_0}{v_0}\,.$$

In particular, the amplitude of oscillation goes like $\mathrm{E}^{1/2}$.
The physical trajectories in the space of states $\mathbb{R}^2$ are ellipses corresponding to the constant energy curves

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \mathrm{E} \geq 0\,,$$

and some of these curves are plotted in Figure 3. In particular, the physical trajectories are closed and the motion is therefore periodic, with period $2\pi/\omega$.

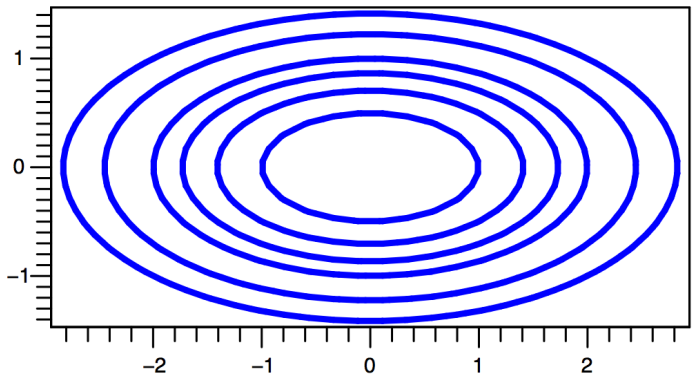


Figure 3: Physical trajectories of equation (16) for $\omega = 2$

2.3 Applications of energy conservation

In this section we show how energy conservation determines the physical trajectories of a one-dimensional conservative mechanical system; that is, one described by equation (15). As remarked above, the energy

$$\mathrm{E} = \tfrac{1}{2} m v^2 + \mathrm{V}(x) \tag{17}$$

is a constant of the motion. Notice that the kinetic energy term ($\frac{1}{2} m v^2$) is always non-negative, therefore $\mathrm{E} \geq \mathrm{V}(x)$ and equality holds if and only if the velocity vanishes; that is, at a turning point. Configurations with potential energy greater than the energy of a particle are inaccessible. In particular, classical particles cannot penetrate potential barriers unless they have sufficient energy. (These statements will be revisited and revised in Quantum Mechanics, which is the subject of the second half of this course.)

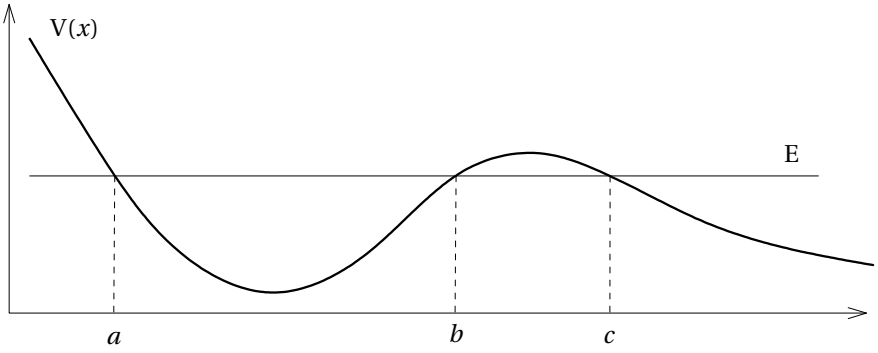


Figure 4: One-dimensional potential motion

Figure 4 illustrates this discussion. It shows the graph of a potential function $\mathrm{V}(x)$ and three turning points: $x = a$, $x = b$ and $x = c$, where $\mathrm{V}(x) = \mathrm{E}$, a fixed value of the energy. Energy conservation means that there are two accessible regions: either the finite interval $[a, b]$ or the semi-infinite interval $[c, \infty)$. If $a \leq x(t) \leq b$ the motion will

be oscillatory and if $x(t) \geq c$ then there are two possibilities: either $\dot{x}(0) < 0$, whence $x(0) > c$ and it will move towards to c and then turn and move away forever, or else $\dot{x}(0) \geq 0$ in which case it will move away from c forever.
In the case of oscillatory motion, we can actually prove that the motion is periodic. This follows from uniqueness of the solution of the initial value problem. Let a be a turning point for a given fixed energy. Then there exists a unique solution $x(t)$ with $x(0) = a$, and hence $\dot{x}(0) = 0$. Now suppose that a certain time T later, $x(\mathrm{T}) = a$ again. The function $x_{\mathrm{T}}(t) := x(t+\mathrm{T})$ solves the same differential equation as x, and $x_{\mathrm{T}}(0) = a$, whence $\dot{x}_{\mathrm{T}}(0) = 0$. By uniqueness, $x_{\mathrm{T}} = x$ and we see that $x(t) = x(t+\mathrm{T})$; that is, x is periodic.
Furthermore, one can use energy conservation to derive an expression for the period. Indeed, from

$$\tfrac{1}{2}m\dot{x}^2 + \mathrm{V}(x) = \mathrm{E}$$

we solve for $\dot{x}$ to obtain

$$\dot{x} = \pm\sqrt{\frac{2(\mathrm{E}-\mathrm{V}(x))}{m}}\,.$$

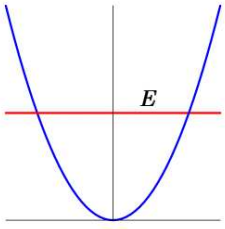
Integrating, we find that the time taken from a to b is

$$\mathrm{T}(a \to b) = \sqrt{\frac{m}{2}}\int_a^b \frac{dx}{\sqrt{\mathrm{E}-\mathrm{V}(x)}}\,. \tag{18}$$

Because of the invariance of Newton's equation (15) under time reversal, this is also the time taken from b to a, whence the period of oscillation is given by

$$\boxed{\mathrm{T} = \sqrt{2m}\int_a^b \frac{dx}{\sqrt{\mathrm{E}-\mathrm{V}(x)}}\,.} \tag{19}$$

Example 2.5 (Simple harmonic motion)**.** Consider Hooke's potential $\mathrm{V}(x) = \frac{1}{2}kx^2$, with $k > 0$.
We saw from the explicit form of the physical trajectories that the period is $2\pi/\omega$ with $\omega^2 = k/m$. Let us rederive this using (19). The limits of integration are the roots of $\mathrm{E} = \frac{1}{2}kx^2$, whence $x = \pm\sqrt{2\mathrm{E}/k}$. Therefore the period is



$$\mathrm{T} = \sqrt{2m}\int_{-\sqrt{\frac{2\mathrm{E}}{k}}}^{\sqrt{\frac{2\mathrm{E}}{k}}} \frac{dx}{\sqrt{\mathrm{E}-\frac{1}{2}kx^2}}\,.$$

Changing variables in the integral to $u = \sqrt{k/2\mathrm{E}}\,x$, we obtain

$$\mathrm{T} = 2\sqrt{\frac{m}{k}}\int_{-1}^{1}\frac{du}{\sqrt{1-u^2}} = 2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\omega}\,,$$

as expected.

The integral in equation (19) has to be treated with care, because the integrand is singular at the limits of integration, since a and b are zeros of $\mathrm{E}-\mathrm{V}(x)$. In fact, it is

not hard to show that the integral converges if and only if a and b are *not* critical points of the potential. To illustrate this, let us suppose that we increase the energy of the particle so that it coincides with a maximum value of the potential, as shown in Figure 5.

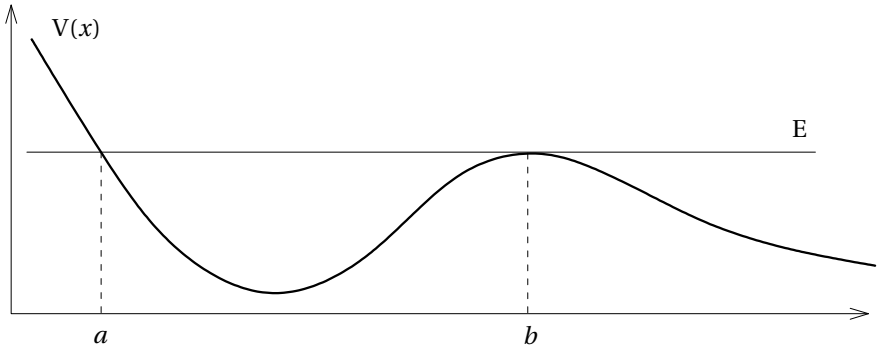


Figure 5: One-dimensional potential motion (cont'd)

Suppose the particle starts from rest at $x(0) = a$. One might be tempted to think that it will move towards b and, upon reaching b, it will turn and come back to a; however this cannot happen, because b is an equilibrium point: if the particle reaches b and turns, it means that it would have zero velocity there, whence it will remain in b forever. What happens in this idealised newtonian universe is that the particle never reaches b! This can be demonstrated by analysing the convergence of the integral which computes the time taken for the particle from a to b, which we now properly write as a limit:

$$\mathrm{T}(a \to b) = \sqrt{\frac{m}{2}} \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} \frac{dx}{\sqrt{\mathrm{E} - \mathrm{V}(x)}} \,.$$

Indeed, let us expand the potential around b to obtain

$$\mathrm{V}(x) = \mathrm{V}(b) + \mathrm{V}'(b)(x-b) + \tfrac{1}{2}\mathrm{V}''(b)(x-b)^2 + \cdots$$

whence if $\mathrm{V}'(b) = 0$, then

$$\mathrm{E} - \mathrm{V}(x) = -\tfrac{1}{2}\mathrm{V}''(b)(b-x)^2 + \cdots \,.$$

The integrand near b is approximated by the first nonzero term in this series expansion. Notice that $\mathrm{V}''(b) \le 0$. If $\mathrm{V}''(b) < 0$, the integral is approximated by

$$2|\mathrm{V}''(b)|^{-1/2} \int \frac{dx}{b-x} \sim -2|\mathrm{V}''(b)|^{-1/2} \log(b-x)$$

which is unbounded as $x \to b$. If $\mathrm{V}''(b) = 0$, then a similar argument shows that the integral behaves like a negative power of $b-x$, which is again unbounded as $x \to b$. In either case, the integral will not converge. In summary, if b is a critical point of the potential, it takes an "infinite" time for the particle to reach b.

Another application of energy conservation, in particular of formula (18), is the determination of the probability of finding the particle in a particular region in space. Let us consider for simplicity the case of oscillatory motion between turning points at $x = a$ and $x = b$ as above. Let $a < c < d < b$ and let us ask the question: what is

the probability of finding the particle in the interval $[c,d]$? What we are after is the **probability density** $\mathrm{P}(x)$ defined in such a way that the

$$\text{probability of finding particle in } [c,d] = \int_c^d \mathrm{P}(x)\,dx\,.$$

On the other hand, the probability of finding the particle in $[c,d]$ is given by the ratio of how long it spends in $[c,d]$ to one period of oscillation. Therefore

$$\begin{aligned}\text{probability of finding particle in } [c,d] &= \frac{\mathrm{T}(c\to d)}{\mathrm{T}(a\to b)}\\ &= \frac{1}{\mathrm{T}(a\to b)}\int_c^d \sqrt{\frac{m}{2}}\,\frac{dx}{\sqrt{\mathrm{E}-\mathrm{V}(x)}}\,,\end{aligned} \tag{20}$$

whence we read off the probability density as

$$\mathrm{P}(x) = \frac{1}{\int_a^b \frac{dy}{\sqrt{\mathrm{E}-\mathrm{V}(y)}}}\,\frac{1}{\sqrt{\mathrm{E}-\mathrm{V}(x)}}\,.$$

Example 2.6 (Harmonic potential)**.** For the harmonic potential $\mathrm{V}(x) = \frac{1}{2}kx^2$, $k>0$, the period $2\mathrm{T}(a\to b) = 2\pi/\omega$, where $\omega^2 = k/m$. Plugging this into equation (20) we obtain

$$\mathrm{P}(x) = \frac{1}{\pi\sqrt{\frac{2\mathrm{E}}{k}-x^2}}\,.$$

This is plotted in Figure 6.

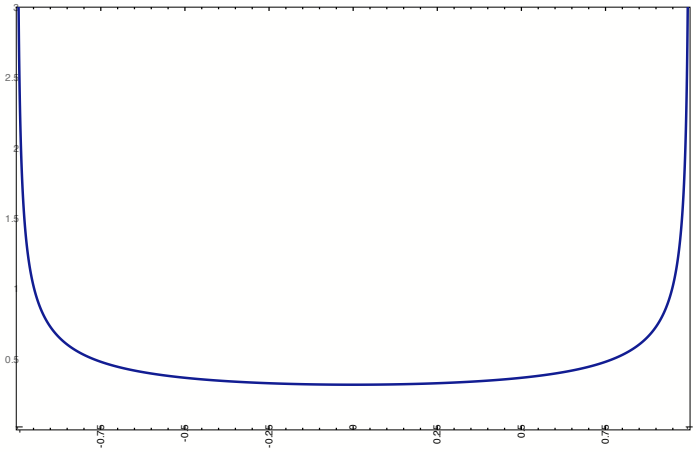


Figure 6: Probability density for harmonic potential and $\mathrm{E} = k/2$

This result for the probability density $\mathrm{P}(x)$ is to be compared with the quantum mechanical probability density $|\Psi(x)|^2$ of a quantum state $\Psi(x)$, to be studied later in the course. You will see (hopefully) that there is a well-defined notion of classical limit in which the quantum probability density $|\Psi(x)|^2$ tends to the classical probability density $\mathrm{P}(x)$. In Quantum Mechanics this is an example of the Correspondence Principle.

Lecture 3: The two-body problem

Eppur si muove. (And yet it does move.)
— Galileo Galilei

In this lecture we will study the mechanics of two massive particles interacting via a conservative force whose potential depends only on the distance between the particles. A special case of such a system is that of a planet orbiting a star, as we will see in the next couple of lectures.

3.1 Centre of mass and the relative problem

Consider two point-particles of masses m_1 and m_2 moving in space subject to a conservative force field whose potential depends only on the distance between the particles. In other words, if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ denote the positions of the particles, the potential depends only on $|\boldsymbol{x}_1 - \boldsymbol{x}_2|$. The configuration space is $\mathbb{R}^6$ and the space of states is therefore $\mathbb{R}^{12}$. Newton's equation are

$$\begin{aligned} m_1 \ddot{\boldsymbol{x}}_1 &= -\nabla_1 V \\ m_2 \ddot{\boldsymbol{x}}_2 &= -\nabla_2 V \,. \end{aligned} \tag{21}$$

We notice that due to the form of the potential, the chain rule implies $\nabla_1 V = -\nabla_2 V$, whence

$$m_1 \ddot{\boldsymbol{x}}_1 + m_2 \ddot{\boldsymbol{x}}_2 = \boldsymbol{0} \,,$$

which in turn implies the conservation of the **centre-of-mass momentum**

$$\boldsymbol{p}_c := m_1 \dot{\boldsymbol{x}}_1 + m_2 \dot{\boldsymbol{x}}_2 \,. \tag{22}$$

It is convenient to change coordinates from $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ to $(\boldsymbol{x}, \boldsymbol{x}_c)$, where $\boldsymbol{x} := \boldsymbol{x}_1 - \boldsymbol{x}_2$ is the **relative coordinate** and

$$\boldsymbol{x}_c := \frac{m_1}{m_1 + m_2} \boldsymbol{x}_1 + \frac{m_2}{m_1 + m_2} \boldsymbol{x}_2 \tag{23}$$

is the **centre-of-mass coordinate**. This is a linear change of coordinates (with unit determinant)

$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{m_1}{m_1+m_2} & \frac{m_2}{m_1+m_2} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} ,$$

which can be easily inverted

$$\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{m_2}{m_1+m_2} & 1 \\ -\frac{m_1}{m_1+m_2} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{pmatrix} .$$

We therefore see that Newton's equation (21) decouples into two equations: one for the centre-of-mass motion,

$$\ddot{\boldsymbol{x}}_c = \boldsymbol{0} \implies \boldsymbol{x}_c(t) = \boldsymbol{x}_c(0) + \frac{\boldsymbol{p}_c t}{m_1 + m_2} \,, \tag{24}$$

and one for the relative motion of the two particles

$$m \ddot{\boldsymbol{x}} = -\nabla V(\boldsymbol{x}) \,, \tag{25}$$

where V depends only on the norm $|\boldsymbol{x}|$, and where we have introduced the **reduced mass**

$$m := \frac{m_1 m_2}{m_1 + m_2} \,. \tag{26}$$

A philosophical observation

Notice that the original physical system of two particles (with masses m_1 and m_2) interacting under a potential depending only on the difference $\boldsymbol{x}_1 - \boldsymbol{x}_2$ of their position vectors has been shown to be *equivalent* to a system consisting of two non-interacting particles: one, a free particle of mass $m_1 + m_2$, and another particle of mass m under the effect of a conservative force field. In the absence of other interactions, no experiment can distinguish between these two cases. Who is to say which of the two descriptions is real? In fact, Physics only *models* Nature, and there is no reason to believe that models are unique and hence we ought to be careful when ascribing elements of truth or reality to our models.

Notation

I hope you will permit a slight abuse of notation. Hereafter we will write $\mathrm{V}(|\boldsymbol{x}|)$ to mean that $\mathrm{V} : \mathbb{R}^3 \to \mathbb{R}$ is a function of $\boldsymbol{x}$ which depends only on $|\boldsymbol{x}|$. A more correct notation would require introducing a function $h : \mathbb{R} \to \mathbb{R}$, say, such that $\mathrm{V}(\boldsymbol{x}) = h(|\boldsymbol{x}|)$. I hope no confusion will result if we omit mentioning this auxiliary function. (Thank you.)

The energy of the two-particle system, which is also conserved, also decomposes into two terms:

$$\begin{aligned} \mathrm{E} &= \tfrac{1}{2} m_1 |\boldsymbol{v}_1|^2 + \tfrac{1}{2} m_2 |\boldsymbol{v}_2|^2 + \mathrm{V}(|\boldsymbol{x}|) \\ &= \frac{1}{2} \frac{|\boldsymbol{p}_c|^2}{m_1 + m_2} + \tfrac{1}{2} m |\boldsymbol{v}|^2 + \mathrm{V}(|\boldsymbol{x}|) \,, \end{aligned}$$

where the first term is the (kinetic) energy of the centre of mass and the last two terms are the (kinetic + potential) energy of the relative motion.
Let us pause to summarise what we have learnt. The physics of these two massive interacting particles is equivalent to the physics of two non-interacting particles: a free particle of mass $m_1 + m_2$ located at the centre of mass of the two original particles and a particle of mass m (the reduced mass) moving under the effect of a conservative force with potential V. Neither of these two particles exist, of course; but their physics is equivalent to the physics of the original system.
Notice that when one of the masses is much larger than the other, say $m_2 \gg m_1$ the reduced mass is very close to the smaller mass m_1 and the centre of mass is very close to $\boldsymbol{x}_2$ and we can approximate this system by one in which the particle of mass m_1 is moving relative to the particle of mass m_2. The next two examples illustrate this for the case of the Earth/Sun system. We will use the Google calculator to perform the calculations.

Example 3.1 (Center of mass of Earth/Sun system)**.** Let us put the Sun at the origin and let the Earth be a distance R away along the x-axis. The centre of mass will be a distance $m_{\oplus}\mathrm{R}/(m_\odot + m_{\oplus})$ along the x-axis, where $m_{\oplus}$ is the mass of the Earth and $m_\odot$ is the mass of the Sun. The average value of R is very close to 1 AU (AU = Astronomical Unit). Typing

```
(mass of earth * 1 AU)/(mass of earth + mass of sun)
```

into Google yields an answer of just under 450 km, which is to be compared with the radius of the Sun, which is approximately 695,500 km!

Example 3.2 (Reduced mass of the Earth/Sun system)**.** The reduced mass is $m = m_{\oplus} m_{\odot}/(m_{\oplus} + m_{\odot})$. Typing

```
(mass of earth * mass of sun)/(mass of earth + mass of sun)
```

into Google yields the answer $5.97418206 \times 10^{24}\,kg$, just under the accepted value of $5.9742 \times 10^{24}\,kg$ for the mass of the Earth (also from Google).

In the next lecture we will study the relative system (25) and show how there is an extra conserved quantity, namely the angular momentum. This will reduce the problem further to an effective one-dimensional problem.

3.2 Elastic collisions

Let us consider the case where the two particles are non-interacting, so that V = 0. In this case not just energy is conserved, but also momentum. Let the particles have momenta $\boldsymbol{p}_1 = m_1 \boldsymbol{v}_1$ and $\boldsymbol{p}_2 = m_2 \boldsymbol{v}_2$, respectively, and energies $\mathrm{E}_1 = \frac{1}{2} m_1 |\boldsymbol{v}_1|^2$ and $\mathrm{E}_2 = \frac{1}{2} m_2 |\boldsymbol{v}_2|^2$, respectively. Now suppose that they undergo an elastic collision, one in which the same two particles emerge but with perhaps different momenta $\boldsymbol{p}'_1 = m_1 \boldsymbol{v}'_1$ and $\boldsymbol{p}'_2 = m_2 \boldsymbol{v}'_2$ and hence different energies $\mathrm{E}'_1 = \frac{1}{2} m_1 |\boldsymbol{v}'_1|^2$ and $\mathrm{E}'_2 = \frac{1}{2} m_2 |\boldsymbol{v}'_2|^2$. The total energy and momentum of the system is given by the sum of the energies and momenta of the individual particles. By momentum conservation,

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}'_1 + \boldsymbol{p}'_2 \tag{27}$$

whereas energy conservation says that

$$\tfrac{1}{2} m_1 |\boldsymbol{v}_1|^2 + \tfrac{1}{2} m_2 |\boldsymbol{v}_2|^2 = \tfrac{1}{2} m_1 |\boldsymbol{v}'_1|^2 + \tfrac{1}{2} m_2 |\boldsymbol{v}'_2|^2 \,. \tag{28}$$

For definiteness we will consider the special case where the second particle is initially at rest, so that $\boldsymbol{v}_2 = \boldsymbol{0}$. The above equations become a little simpler:

$$\boldsymbol{p}_1 = \boldsymbol{p}'_1 + \boldsymbol{p}'_2$$

and

$$\tfrac{1}{2} m_1 |\boldsymbol{v}_1|^2 = \tfrac{1}{2} m_1 |\boldsymbol{v}'_1|^2 + \tfrac{1}{2} m_2 |\boldsymbol{v}'_2|^2 \,.$$

The momenta before and after the collision are depicted in Figure 7, from where it is clear that the motion takes place in the plane spanned by the final momenta.
The momentum conservation equation (27) has two components: one parallel and one perpendicular to $\boldsymbol{p}_1$, which give rise to two equations

$$\begin{aligned} m_1 |\boldsymbol{v}_1| &= m_1 |\boldsymbol{v}'_1| \cos\theta_1 + m_2 |\boldsymbol{v}'_2| \cos\theta_2 \\ 0 &= m_1 |\boldsymbol{v}'_1| \sin\theta_1 - m_2 |\boldsymbol{v}'_2| \sin\theta_2 \,. \end{aligned}$$

Together with energy conservation, there are a total of three equations for four unknowns: $|\boldsymbol{v}'_1|$, $|\boldsymbol{v}'_2|$, θ_1 and θ_2, so we will not be able to determine the motion uniquely.

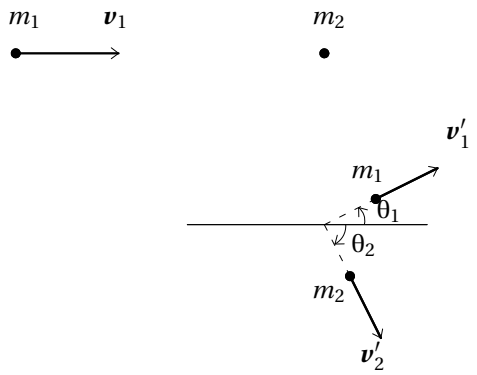


Figure 7: Elastic collision

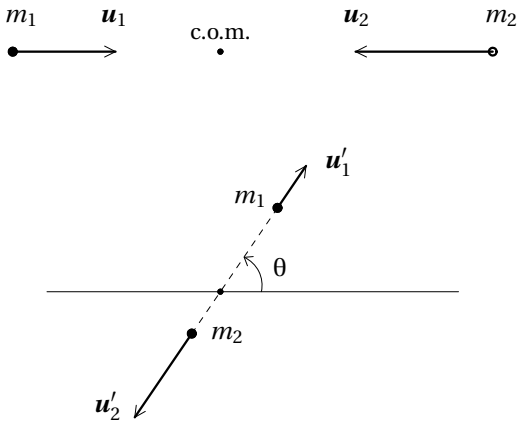


Figure 8: Elastic collision relative to the centre of mass

A typical question we might hope to answer is, for example, "What is the maximum possible value of θ_1?"
It is easier to answer this question by studying the motion relative to the centre of mass, as illustrated in Figure 8.
The centre-of-mass velocity $\boldsymbol{v}_c$ is constant and equal to

$$\boldsymbol{v}_c = \frac{m_1}{m_1+m_2}\boldsymbol{v}_1 \,,$$

where we have used that $\boldsymbol{v}_2 = \boldsymbol{0}$. The velocities relative to the centre of mass are given by

$$\begin{aligned} \boldsymbol{u}_1 &= \boldsymbol{v}_1 - \boldsymbol{v}_c \\ \boldsymbol{u}_2 &= -\boldsymbol{v}_c \end{aligned} \qquad \text{and} \qquad \begin{aligned} \boldsymbol{u}_1' &= \boldsymbol{v}_1' - \boldsymbol{v}_c \\ \boldsymbol{u}_2' &= \boldsymbol{v}_2' - \boldsymbol{v}_c \,, \end{aligned}$$

again using that $\boldsymbol{v}_2 = \boldsymbol{0}$. Relative to the centre-of-mass, motion before and after the collision is collinear, for otherwise the centre-of-mass would not remain at rest. Indeed,

$$m_1\boldsymbol{u}_1 + m_2\boldsymbol{u}_2 = \boldsymbol{0} \qquad \text{and} \qquad m_1\boldsymbol{u}_1' + m_2\boldsymbol{u}_2' = \boldsymbol{0}\,.$$

To see this, simply write $\boldsymbol{v}_1 = \boldsymbol{u}_1 + \boldsymbol{v}_c$ and $\boldsymbol{v}_2 = \boldsymbol{u}_2 + \boldsymbol{v}_c$ and the same for the primed velocities after the collision. Inserting these expressions in the definition of the centre-

of-mass velocity $\boldsymbol{v}_c$

$$(m_1+m_2)\,\boldsymbol{v}_c = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = m_1\boldsymbol{v}_1' + m_2\boldsymbol{v}_2' \,,$$

we obtain the desired equations.
Similarly, energy conservation says that

$$\tfrac{1}{2}m_1|\boldsymbol{u}_1|^2 + \tfrac{1}{2}m_2|\boldsymbol{u}_2|^2 = \tfrac{1}{2}m_1|\boldsymbol{u}_1'|^2 + \tfrac{1}{2}m_2|\boldsymbol{u}_2'|^2 \,,$$

where we have subtracted the centre-of-mass energy from both sides. Using the above results we can express the velocities of the second particle in terms of those of the first, and the resulting equation says that $|\boldsymbol{u}_1| = |\boldsymbol{u}_1'|$, which when re-inserted in the energy conservation equation yields that $|\boldsymbol{u}_2| = |\boldsymbol{u}_2'|$.
In other words, relative to the centre of mass, the velocities get rotated by an angle θ, as shown in Figure 8. However, energy and momentum conservation do not constrain this angle further.
Let us now relate the angles θ_1 and θ. Consider the equation $\boldsymbol{v}_1' = \boldsymbol{u}_1' + \boldsymbol{v}_c$ and let us look at the components parallel and perpendicular to the centre-of-mass velocity:

$$\begin{aligned} (\parallel): &\quad |\boldsymbol{v}_1'|\cos\theta_1 = |\boldsymbol{u}_1'|\cos\theta + |\boldsymbol{v}_c| \\ (\perp): &\quad |\boldsymbol{v}_1'|\sin\theta_1 = |\boldsymbol{u}_1'|\sin\theta \,, \end{aligned}$$

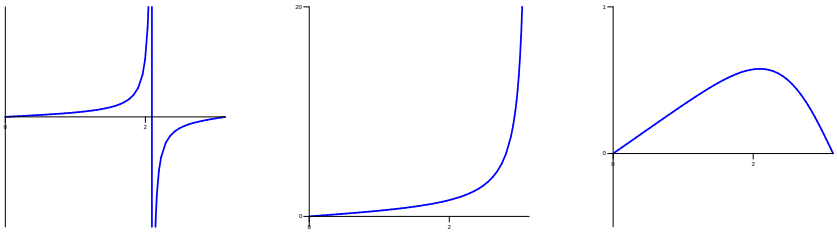
whence

$$\tan\theta_1 = \frac{|\boldsymbol{v}_1'|\sin\theta_1}{|\boldsymbol{v}_1'|\cos\theta_1} = \frac{|\boldsymbol{u}_1|\sin\theta}{|\boldsymbol{u}_1|\cos\theta + |\boldsymbol{v}_c|} \,,$$

where we have used that $|\boldsymbol{u}_1| = |\boldsymbol{u}_1'|$. Since $\boldsymbol{v}_2 = 0$, it follows that $|\boldsymbol{v}_c| = |\boldsymbol{u}_2|$, whence dividing top and bottom by $|\boldsymbol{u}_1|$ and using that $|\boldsymbol{u}_2|/|\boldsymbol{u}_1| = m_1/m_2$, we obtain

(29)
$$\boxed{\tan\theta_1 = \frac{\sin\theta}{\cos\theta + m_1/m_2} \,.}$$

In Figure 9 we have sketched $\tan\theta_1$ as a function of $\theta \in [0,\pi]$ distinguishing between three cases, according to whether m_1/m_2 is smaller than, equal to or greater than 1.



(a) $m_1 < m_2$ (b) $m_1 = m_2$ (c) $m_1 > m_2$

Figure 9: $\tan\theta_1$ as a function of θ for different masses

In the first case, when $m_1 < m_2$ we see that all angles $\theta_1 \in [0,\pi]$ are possible, with $\tan\theta_1$ becoming unbounded both below and above at $\theta_0 = \cos^{-1}(-m_1/m_2) > \frac{\pi}{2}$.
In the second case, when $m_1 = m_2$ all angles $\theta_1 \in [0,\frac{\pi}{2}]$ are possible, with $\tan\theta_1$ becoming unbounded above at $\theta = \pi$. Finally, the most interesting case is when $m_1 > m_2$, where there is a maximum value for θ_1.

Example 3.3 (The case $m_1 > m_2$)**.** When $m_1 > m_2$, $\tan\theta_1$ has a maximum at $\theta_0 = \cos^{-1}(-m_2/m_1) > \frac{\pi}{2}$ and its maximum value is

$$\theta_{1\max} = \sin^{-1}(m_2/m_1) \le \frac{\pi}{2}\,.$$

Indeed, differentiating $\tan\theta_1$ with respect to θ, we find

$$\frac{d}{d\theta}\tan\theta_1 = \frac{1+(m_1/m_2)\cos\theta}{(\cos\theta + m_1/m_2)^2}\,,$$

whence the maximum occurs at θ_0, where $\cos\theta_0 = -m_2/m_1$. At this value of θ, we find

$$\tan\theta_{1\max} = \frac{1}{\sqrt{(m_1/m_2)^2-1}} = \frac{m_2/m_1}{\sqrt{1-(m_2/m_1)^2}} = \frac{m_2}{\sqrt{m_1^2-m_2^2}}\,,$$

which can be inverted to obtained the above value for $\theta_{1\max}$.

Lecture 4: Central force fields

What makes the planets go around the sun? At the time of Kepler some people answered this problem by saying that there were angels behind them beating their wings and pushing the planets around their orbit. This answer is not very far from the truth. The only difference is that the angels sit in a different direction and their wings push inwards.

— Richard Feynman

In the last lecture we saw how the two-body problem decoupled into the free motion of the centre of mass and a relative problem governed by equation (25):

$$m\ddot{\boldsymbol{x}} = -\nabla \mathrm{V}(|\boldsymbol{x}|)\,,$$

where m is the reduced mass (26). In this lecture we will study this equation.

4.1 Central forces

A force field $\mathbf{F}$ is called **central** if

$$\mathbf{F}(\boldsymbol{x}) = f(\boldsymbol{x})\boldsymbol{x} \tag{30}$$

for some function $f : \mathbb{R}^3 \to \mathbb{R}$; that is, if it points in the direction of the line through $\boldsymbol{x}$, for all $\boldsymbol{x}$. It is often the case that such a force field is singular at the origin, and if this occurs we will implicitly restrict the configuration space to those points with $|\boldsymbol{x}| > 0$. Using that

$$\nabla|\boldsymbol{x}| = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\,,$$

we see that the force field $\mathbf{F} = -\nabla \mathrm{V}$ in equation (25) is given by

$$\mathbf{F} = -\frac{\mathrm{V}'(|\boldsymbol{x}|)}{|\boldsymbol{x}|}\boldsymbol{x}\,, \tag{31}$$

whence it is central.
We will now prove that for a conservative field, the property of being central can be characterised in other ways. Indeed, the following are equivalent for a conservative force field $\mathbf{F}$:

(a) $\mathbf{F}$ is central,

(b) $\mathbf{F} = f(|\boldsymbol{x}|)\boldsymbol{x}$,

(c) $\mathbf{F} = -\nabla \mathrm{V}(|\boldsymbol{x}|)$.

Indeed, we have already seen that (c) implies (b), and by definition (b) implies (a). It remains to show that (a) implies (c). This is equivalent to showing that V is constant in each sphere $\mathrm{S_R} = \{\boldsymbol{x} \in \mathbb{R}^3 \mid |\boldsymbol{x}| = \mathrm{R}\}$, so it only depends on $|\boldsymbol{x}|$ and not on its direction. We first observe (prove it!) that any two points in $\mathrm{S_R}$ can be joined by a path on $\mathrm{S_R}$. This reduces the problem to proving that V does not change along any path on $\mathrm{S_R}$. Let $\boldsymbol{x}(t)$ be a path on $\mathrm{S_R}$. This means that $|\boldsymbol{x}(t)| = \mathrm{R}$ for all t. Differentiating

with respect to t we learn that $\boldsymbol{x}(t)\cdot\dot{\boldsymbol{x}}(t)=0$ for all t. Now let $\mathrm{V}(\boldsymbol{x}(t))$ be the value of V along this path. Differentiating with respect to t and using (14), we see that

$$\frac{d}{dt}\mathrm{V}(\boldsymbol{x}(t))=\nabla\mathrm{V}\big|_{\boldsymbol{x}(t)}\cdot\dot{\boldsymbol{x}}(t)=-\mathbf{F}(\boldsymbol{x}(t))\cdot\dot{\boldsymbol{x}}(t)\,.$$

For a central field, $\mathbf{F}(\boldsymbol{x})=f(\boldsymbol{x})\boldsymbol{x}$, whence

$$\frac{d}{dt}\mathrm{V}(\boldsymbol{x}(t))=-f(\boldsymbol{x}(t))\boldsymbol{x}(t)\cdot\dot{\boldsymbol{x}}(t)=0\,,$$

as expected.
In other words, we learn that potentials for central force fields have the property that they only depend on the length $|\boldsymbol{x}|$ and not on the direction. This means that they are constant on the spheres of constant $|\boldsymbol{x}|$. For this reason such potentials are said to be **spherically symmetric**; although a better name would be spherically constant!

4.2 Conservation of angular momentum

An important property of central force fields—even if not conservative—is that motion is restricted to a plane, namely the plane spanned by $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$. In fact, more is true. Let us define the **angular momentum** vector

$$\mathbf{L}=\boldsymbol{x}\times\boldsymbol{p}\,,\qquad\text{with}\quad\boldsymbol{p}=m\dot{\boldsymbol{x}}\,.$$

Explicitly, $\mathbf{L}=\begin{pmatrix}\mathrm{L}_1\\ \mathrm{L}_2\\ \mathrm{L}_3\end{pmatrix}$, where

$$\mathrm{L}_1=x_2p_3-x_3p_2\qquad\mathrm{L}_2=x_3p_1-x_1p_3\qquad\mathrm{L}_3=x_1p_2-x_2p_1\,,\tag{32}$$

where $\boldsymbol{x}=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$ and $\boldsymbol{p}=\begin{pmatrix}p_1\\ p_2\\ p_3\end{pmatrix}$. The angular momentum has dimension of $[\mathbf{L}]=\mathrm{ML}^2\mathrm{T}^{-1}$.
Whereas $\boldsymbol{x}$ and $\boldsymbol{p}$ evolve in time, for a central force field $\mathbf{L}$ is constant. Indeed, using the product rule

$$\frac{d}{dt}\mathbf{L}=\dot{\boldsymbol{x}}\times\boldsymbol{p}+\boldsymbol{x}\times\mathbf{F}\,,$$

where we have used Newton's equation in the form $\frac{d}{dt}\boldsymbol{p}=\mathbf{F}$. Now $\dot{\boldsymbol{x}}$ and $\boldsymbol{p}$ are parallel whence the first term vanishes, and for a central force field $\mathbf{F}$ and $\boldsymbol{x}$ are also parallel, so the second term vanishes as well.
If $\mathbf{L}\neq\mathbf{0}$ then it defines a perpendicular plane, which is the plane spanned by $\boldsymbol{x}$ and $\boldsymbol{p}=m\dot{\boldsymbol{x}}$. If $\mathbf{L}=\mathbf{0}$, then $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are parallel, but then so is $\ddot{\boldsymbol{x}}$, whence the motion is linear, which is a particular case of planar motion.
Conservation of angular momentum has reduced a three-dimensional problem to a two-dimensional problem, but in fact we can do better. This is because in restricting to the plane we have only used that the direction of the angular momentum is constant. The fact that also the magnitude of the angular momentum is conserved will reduce this problem to an effective one-dimensional problem which we now describe.

The ϵ-symbol

In the Physics notation introduced in Section 1.1, the angular momentum L_i can be written in terms of the ϵ-symbol as

$$\mathrm{L}_i = \epsilon_{ijk} x_j p_k \,,$$

where you are reminded that repeated indices are summed over. The symbol ϵ_{ijk} is uniquely defined by the following two properties. First, it is **alternating**, so that under permutation of the labels it gets multiplied by the sign of the permutation; e.g.,

$$\epsilon_{132} = -\epsilon_{123} \,, \quad \epsilon_{231} = \epsilon_{123} \,, \quad \text{etc}$$

In particular, it vanishes whenever two of the labels agree. Last, it is normalised so that $\epsilon_{123} = +1$. The first equation in (32) can thus be recovered as follows

$$\begin{aligned} \mathrm{L}_1 &= \epsilon_{1jk} x_j p_k \\ &= \epsilon_{123} x_2 p_3 + \epsilon_{132} x_3 p_2 \\ &= x_2 p_3 - x_3 p_2 \,, \end{aligned}$$

and similarly for the other components.

4.3 The effective one-dimensional problem

Let us reorient our axes so that $\mathbf{L} = \begin{pmatrix} 0 \\ 0 \\ m\ell \end{pmatrix}$ is pointing along the x_3-axis, so that the motion is restricted to the (x_1, x_2)-plane. It is convenient to employ polar coordinates (r, θ) so that

$$x_1 = r\cos\theta \qquad \text{and} \qquad x_2 = r\sin\theta \,.$$

The momentum vector $\boldsymbol{p}$ has (nonzero) components

$$\begin{aligned} p_1 &= m\dot{x}_1 = m\dot{r}\cos\theta - mr\dot{\theta}\sin\theta \\ p_2 &= m\dot{x}_2 = m\dot{r}\sin\theta + mr\dot{\theta}\cos\theta \,. \end{aligned}$$

Therefore conservation of angular momentum, which says that $\mathrm{L}_3 = m\ell = x_1 p_2 - x_2 p_1$ is constant, becomes

$$\boxed{r^2\dot{\theta} = \ell} \,. \tag{33}$$

Notice that if $\ell \neq 0$, then $\dot{\theta} \neq 0$ and it never changes sign: it is either always positive or always negative.

Kepler's area law

Equation (33) has a geometrical interpretation. Indeed, $\frac{1}{2}r^2\dot{\theta}$ is the **areal velocity**; that is, the rate of change of the area swept by the particle as it moves, as illustrated in Figure 10. Conservation of angular momentum says therefore not just that motion is planar, but that the areal velocity is constant, whence equal areas are swept in equal time. This is Kepler's second law of planetary motion; although now we understand that this law is more general and applies to any central force field.

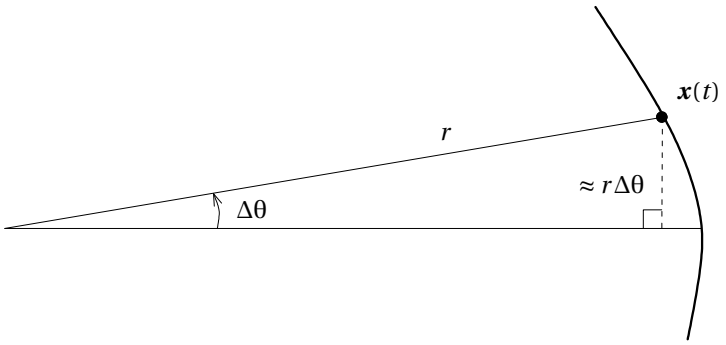


Figure 10: Area swept during motion in central force field

Equation (33) can be used to solve for $\dot{\theta}$ in terms of r, effectively eliminating this variable from Newton's equation, which thus reduces to an equation for r. Indeed, let us again assume that $\mathbf{F}$ is conservative with potential V. Then from equation (31), using that $|\boldsymbol{x}| = r$, we see that

$$\mathbf{F} = -\mathrm{V}'(r)\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix},$$

whereas

$$\frac{d}{dt}\boldsymbol{p} = m\left(\ddot{r} - \frac{\ell^2}{r^3}\right)\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix},$$

where we have used equation (33) to get rid of any derivatives of θ. Finally, Newton's equation $\frac{d}{dt}\boldsymbol{p} = \mathbf{F}$ becomes an ODE for r

(34) $$\boxed{m\ddot{r} = -\mathrm{V}'(r) + \frac{m\ell^2}{r^3}},$$

which can be written as

(35) $$m\ddot{r} = -\mathrm{V}'_{\mathrm{eff}}(r),$$

where we have introduced an **effective potential**

$$V_{\text{eff}} = V + \frac{m\ell^2}{2r^2} \quad . \tag{36}$$

Equivalently, without reference to any choice of orientation of the axes, we can write the effective potential also as

$$V_{\text{eff}} = V + \frac{|\mathbf{L}|^2}{2m|\boldsymbol{x}|^2} \ .$$

Energy conservation for the effective one-dimensional problem says that

$$E = \tfrac{1}{2}m\dot{r}^2 + V + \frac{m\ell^2}{2r^2} \tag{37}$$

is a constant. The last term is called the **centrifugal energy** and is the kinetic energy due to the angular velocity: $\frac{1}{2}mr^2\dot{\theta}^2$. We can analyse the physics of this system as we did in Section 2.3, keeping in mind one important difference. The turning points where $E = V_{\text{eff}}$ are now *not* points of zero velocity, since the angular velocity is nonzero for nonzero angular momentum. Instead they are simply local minima or local maxima of the function $r(t)$.

We can distinguish between two different kinds of motion, depending on whether the energy condition restricts r to obey $r \geq r_{\min}$ or to obey $r_{\max} \geq r \geq r_{\min}$. In the former case, the particle comes from and returns to infinity, or in two-body language, the two bodies are infinitely far apart in the remote past and will go back to being infinitely far apart in the remote future: a physical process called **scattering**. In the latter case, the path of the particle will lie within the annulus bounded by $r_{\max}$ and $r_{\min}$, or in two-body language, the distance between the two bodies will oscillate forever between $r_{\max}$ and $r_{\min}$. Unlike the case of truly one-dimensional motion, this does not mean however that the orbits are closed. This is because θ is also evolving, so going back to $r_{\max}$, say, does not mean going back to the same point in the plane of motion. This is illustrated in Figure 11.

To investigate when the orbits are closed, let us derive an expression for the angle $\Delta\theta$ through which $\boldsymbol{x}(t)$ turns in the time it takes for $r(t)$ to go from $r_{\min}$ to $r_{\max}$ and back, as shown in Figure 11. From equation (37),

$$\dot{r} = \pm\sqrt{\frac{2}{m}\left(E - V(r)\right) - \frac{\ell^2}{r^2}} \ ,$$

whereas from equation (33),

$$\dot{\theta} = \frac{\ell}{r^2} \ .$$

These two expressions yield a formula for $\frac{d\theta}{dr}$:

$$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \pm\frac{\ell/r^2}{\sqrt{\frac{2}{m}\left(E - V(r)\right) - \frac{\ell^2}{r^2}}} \ .$$

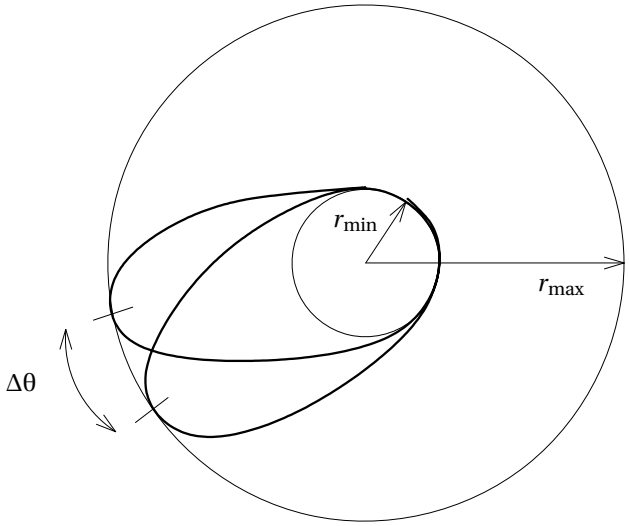


Figure 11: Turning angle for bounded orbits

Invariance under time-reversal means that it takes just as long to go from $r_{\max}$ to $r_{\min}$ than from $r_{\min}$ back to $r_{\max}$, whence

(38)
$$\Delta\theta = 2\int_{r_{\min}}^{r_{\max}} \frac{\ell\, dr/r^2}{\sqrt{\frac{2}{m}(E - V(r)) - \frac{\ell^2}{r^2}}}\,.$$

Example 4.1 (Periodicity of orbits)**.** We will show that the motion is periodic if and only if $\Delta\theta$ is a rational multiple of 2π; that is, $\Delta\theta = \frac{2m}{n}\pi$ for (coprime) integers m and n. To see this we observe that by uniqueness of solutions to ODEs, the motion will be periodic if and only if the particle passes twice by the same point $(r_{\max}, \theta_0)$. Indeed, when $r = r_{\max}$, $\dot r = 0$ and $\dot\theta = \ell/r_{\max}^2$. And the value of θ_0 fixes the solution uniquely. From Figure 11 it is clear that this will happen if and only if for some positive integer n, $n\Delta$ is an integer multiple of 2π.

There are two potentials $V(r)$ for which (bounded) motion is always periodic: r^2 and r^{-1}. The latter potential is the one responsible for planetary motion and will be studied in detail in the next lecture.

Example 4.2 (Periodic orbits in the space oscillator)**.** For the space oscillator potential $V(r) = \frac{1}{2}kr^2$, with $k > 0$, $\Delta\theta = \pi$ for $\ell > 0$ and any (admissible) value of the other parameters and hence the orbits are periodic. To prove this takes a little bit of calculation.
We compute $\Delta\theta$ from equation (38), for which we need to determine the minimum and maximum radii, which are the roots of the equation

$$E = \tfrac{1}{2}kr^2 + \frac{m\ell^2}{2r^2}\,,$$

which are found to be

$$r_{\min}^2 = \frac{E}{k}\left(1-\sqrt{1-\frac{km\ell^2}{E^2}}\right) \qquad \text{and} \qquad r_{\max}^2 = \frac{E}{k}\left(1+\sqrt{1-\frac{km\ell^2}{E^2}}\right).$$

In terms of these, the integral becomes

$$\Delta\theta = 2\sqrt{\frac{m\ell^2}{k}}\int_{r_{\min}}^{r_{\max}} \frac{dr/r}{\sqrt{(r^2-r_{\min}^2)(r_{\max}^2-r^2)}}\,.$$

We now embark in a sequence of changes of variables until we can reach an integral we can evaluate. We define $u := kr^2/E$ and introduce the parameter $\tau = 1-\sqrt{\frac{mk\ell^2}{E^2}}$, in terms of which the turning angle becomes

$$\Delta\theta = \sqrt{1-\tau}\int_{1-\sqrt{\tau}}^{1+\sqrt{\tau}} \frac{du/u}{\sqrt{(u-1+\sqrt{\tau})(1+\sqrt{\tau}-u)}}\,.$$

Let $v = \frac{\sqrt{u-1+\sqrt{\tau}}}{\sqrt{2\sqrt{\tau}}}$, in terms of which

$$\Delta\theta = 2\sqrt{1-\tau}\int_0^1 \frac{dv}{\sqrt{1-v^2}}\frac{1}{1+\sqrt{\tau}(2v^2-1)}\,.$$

Now let $w = \frac{v}{\sqrt{1-v^2}}$, so that

$$\Delta\theta = 2\sqrt{1-\tau}\int_0^\infty \frac{dw}{1+w^2}\frac{1}{1+\sqrt{\tau}\frac{w^2-1}{1+w^2}}\,,$$

which can be rewritten as

$$\begin{aligned}\Delta\theta &= 2\sqrt{1-\tau}\int_0^\infty \frac{dw}{1-\sqrt{\tau}+(1+\sqrt{\tau})w^2}\\ &= 2\sqrt{\frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}}\int_0^\infty \frac{dw}{\frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}+w^2}\,.\end{aligned}$$

Writing $\alpha = \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}} > 0$ and letting $z = w/\alpha$, we arrive at an elementary integral we can evaluate, namely

$$\Delta\theta = 2\int_0^\infty \frac{dz}{1+z^2} = \pi\,.$$

Summarising what we have learnt in the last two lectures, we have reduced the two-body problem to the problem of a particle on the half-line ($r>0$) subject to an effective potential obtained by adding to the original potential a term involving the angular momentum. To solve the two-body problem we must then solve this problem first for $r(t)$, we then use (33) to solve for θ and hence solve for the relative motion of the two bodies. Finally we add the motion of the centre of mass.

Lecture 5: The Kepler and Coulomb problems

If I let go of a hammer on a planet of positive gravity, I need not see it fall to know that it has, in fact, fallen.

— Spock, stardate 2948.9

Newton's universal law of gravitation states that two massive bodies—e.g., hammer and planet, Earth and Sun—exert an attractive force on each other which is proportional to the product of their masses and inverse proportional to the square of the distance separating them. Newton showed that this law recovered galilean gravity in a certain regime (appropriate to the dynamics of falling objects near the surface of the Earth) but also recovered Kepler's laws on planetary motion. It was the first truly modern physical theory and it proved unassailable for more than two centuries, when it was replaced (for now) by Einstein's theory of general relativity.

5.1 The Kepler problem

A more precise mathematical statement of this law is the following. The gravitational potential between two point-particles of masses m_1 and m_2 at positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively is given by

$$V = -\frac{G m_1 m_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\,, \tag{39}$$

where $G \approx 6.67300 \times 10^{-11}\, m^3 kg^{-1} s^{-2}$ is **Newton's constant**. The first precise measurement of this constant was done by Cavendish.
How about for a body which is not a point particle? Let a massive body occupy a compact—that is, closed and bounded—subset $S \subset \mathbb{R}^3$. The density function of the body is a positive function $\mu : S \to \mathbb{R}$ with the property that the volume integral of μ on S is equal to the mass m_1 of the body. By definition, the gravitational potential felt by a point particle of mass m_2 at a point $\boldsymbol{x}$ outside S is given by the volume integral

$$V(\boldsymbol{x}) := -G m_2 \int_S \frac{\mu(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} d^3\boldsymbol{y}\,. \tag{40}$$

In the case of planetary motion (or even falling apples), if we model the bodies in question as spheres of uniform density, then the gravitational potential coincides with the one of a point-particle sitting at the centre of mass. The next example illustrates this.

Example 5.1 (Gravitational potential of a uniform sphere)**.** Consider a point-particle of mass m_2 a distance r from the centre of a uniform sphere of mass m_1 and radius R, a situation is illustrated by Figure 12. Let us use spherical polar coordinates (ρ, θ, φ) in which the volume element is $\rho^2 \sin\theta d\rho d\theta d\varphi$. The distance from the point-particle to a point in the sphere with coordinates (ρ, θ, φ) is denoted l in the picture. Pythagoras tells us that

$$l^2 = (\rho \sin\theta)^2 + (r - \rho\cos\theta)^2 = \rho^2 + r^2 - 2\rho r \cos\theta\,.$$

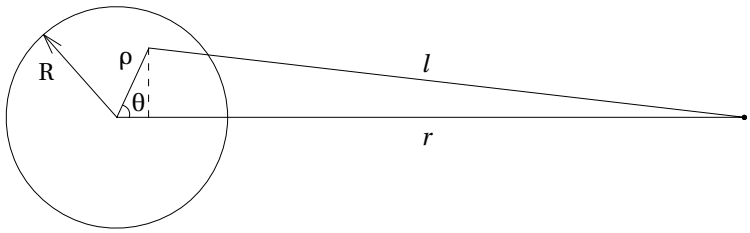


Figure 12: Gravitational potential of a uniform sphere

For a uniform body the density is constant, whence equation (40) becomes

$$V = -Gm_2 \int \frac{\mu\rho^2 \sin\theta d\rho d\theta d\varphi}{\sqrt{\rho^2 + r^2 - 2\rho r\cos\theta}} .$$

Nothing depends on φ, so its integral gives 2π. Next we do the θ integral by changing variables to $u = \cos\theta$, yielding

$$V = -2\pi\mu G m_2 \int_{\rho=0}^{R}\int_{u=-1}^{1} \frac{\rho^2 d\rho du}{\sqrt{\rho^2 + r^2 - 2\rho r u}} .$$

Performing the elementary u-integral we get

$$V = -\frac{4\pi\mu G m_2}{r}\int_0^R d\rho\rho^2$$

and performing the ρ-integral and recognising $m_1 = 4\pi\mu R^3/3$, we get

$$V = -\frac{Gm_1 m_2}{r} \qquad \text{for } r > R.$$

5.2 The galilean limit

We will first show that newtonian gravity reduces to galilean gravity; so that, for example, on the surface of the Earth the force of gravity can be taken to be constant. From Example 5.1, the force of gravity due to the Earth is the same as if all the mass of the Earth were concentrated at its centre of mass. We are looking at displacements (e.g., apples falling from medieval buildings) which are small compared from the distance to the centre of the Earth. This is illustrated in Figure 13.
According to equation (39), the gravitational potential felt by the apple (of mass m) at a height z above the surface of the Earth (of mass M) is given by

$$V(z) = -\frac{GMm}{R+z} = -\frac{GMm}{R}\frac{1}{1+z/R} \approx -\frac{GMm}{R} + \frac{GMm}{R^2}z + O\left((z/R)^2\right)$$

Ignoring the irrelevant constant term, we see that to first order in z/R, we obtain the galilean potential

$$V(z) = mgz\,, \qquad \text{where} \quad g = \frac{GM}{R^2} .$$

For example, typing

```
(G * mass of earth)/(radius of earth)^2
```

into Google produces the answer $g \approx 9.79982305\ ms^{-2}$. (Before you get too excited, let me point out that this argument is circular, since to weigh the Earth one uses this formula backwards from an empirical knowledge of g.)

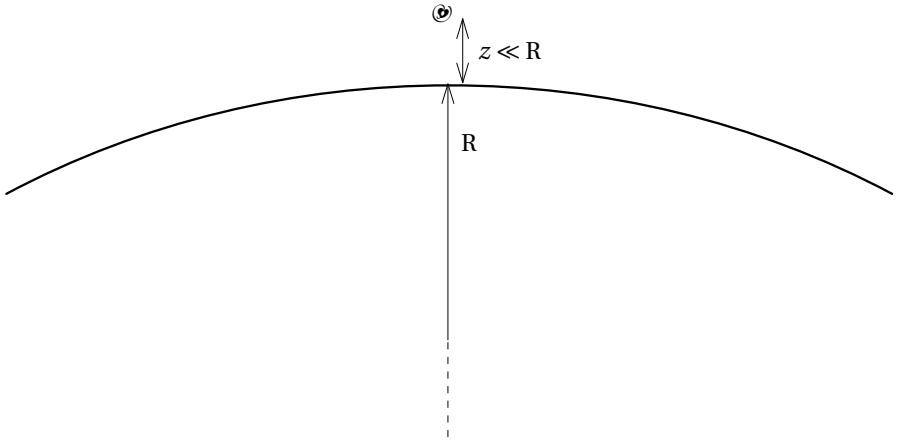


Figure 13: The galilean limit of newtonian gravity

5.3 The Coulomb potential

Our experience to this (star)date suggests that the gravitational potential is always attractive: objects simply fall. However mathematically there is nothing which prevents us from changing the sign of the potential and consider a repulsive force: a sort of 'negative' gravity. Interestingly such a potential does exist in nature: it is the electrostatic potential between charged particles, also known as the Coulomb potential. The potential is formally the same as in the case of gravity, except that instead of masses we have charges, and this is a crucial difference because unlike masses, charges come in two flavours: positive and negative. The Coulomb potential is usually written as

$$\mathrm{V} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\,, \tag{41}$$

which is repulsive if the charges are of the same sign and attractive otherwise. The constant ε_0 is called the electrical permittivity (here, of empty space), but this shall not matter in this course.
Much of what we will say will hold regardless of the constant in front of the potential, whether it is positive of negative, so hereafter we will consider the case of a potential of the form

$$\mathrm{V} = -\frac{k}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\,, \tag{42}$$

where k is a constant which depends on the Physics we are modelling, and has dimension $[k] = \mathrm{ML}^3\mathrm{T}^{-2}$. It is positive and equal to $\mathrm{G}m_1 m_2$ in the case of gravitational interactions and of indefinite sign and equal to $-q_1 q_2/4\pi\varepsilon_0$ in the case of electrostatic interactions.

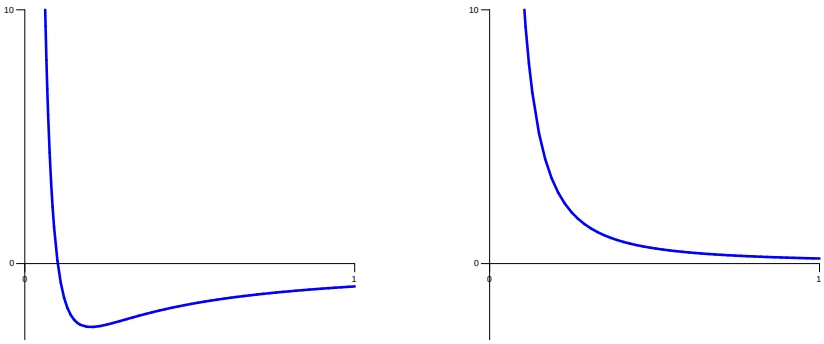
5.4 The effective potential

The potential (42) is an example of the two-body problem discussed in Lecture 3. After decoupling the centre-of-mass motion, we are left with an effective one-dimensional

system describing a particle of (reduced) mass $m = m_1 m_2/(m_1+m_2)$ in the presence of an effective potential

$$V_{\text{eff}} = -\frac{k}{r} + \frac{m\ell^2}{2r^2}\,. \tag{43}$$

This potential is sketched in Figure 14 for $\ell \neq 0$ for both signs of k. In particular the energy is always positive in the repulsive case, whence only in the attractive case can we have bounded orbits.



(a) attractive ($k > 0$) (b) repulsive ($k < 0$)

Figure 14: Effective potentials for Kepler and Coulomb problems

Example 5.2 (Minimum radius)**.** The function $V(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2}$, for positive α and β, has a minimum at the roots of $V'(r) = 0$. The only root is $r = 2\beta/\alpha$, where V takes the value $-\alpha^2/4\beta$. Inserting the values of $\alpha = k$ and $\beta = \frac{1}{2}m\ell^2$ we obtain that the effective potential V_{eff} has a minimum at $r = m\ell^2/k$ with value $V_{\min} = -k^2/2m\ell^2$.

It follows from energy conservation that if $E \geq 0$ the motion is unbounded, whereas if $E < 0$ the motion is constrained to lie in the annulus $r_{\max} \geq r \geq r_{\min}$, where

$$\frac{1}{r_{\min}} = \frac{k}{m\ell^2}\left(1 + \sqrt{1 - \frac{2|E|m\ell^2}{k^2}}\right) \tag{44}$$

$$\frac{1}{r_{\max}} = \frac{k}{m\ell^2}\left(1 - \sqrt{1 - \frac{2|E|m\ell^2}{k^2}}\right)$$

For planets going around the Sun, $r_{\min}$ is called the **aphelion** and $r_{\max}$ the **perihelion**. For the moon (or a satellite) going around the Earth, these are called **apogee** and **perigee**, et cetera.

Example 5.3 (Periodicity of bounded orbits in the Kepler problem)**.** Let $E < 0$ and $\ell \neq 0$. The turning angle $\Delta\theta$ is given by equation (38) where $r_{\min}$ and $r_{\max}$ are given

in equation (44). After dividing the integrand by ℓ, we find

$$\Delta\theta = \pm 2 \int_{r_{\min}}^{r_{\max}} \frac{dr/r^2}{\sqrt{\left(\frac{1}{r} - \frac{1}{r_{\max}}\right)\left(\frac{1}{r_{\min}} - \frac{1}{r}\right)}}$$

where the sign refers to the sign of ℓ. Let us take $\ell > 0$ from now on. Let us change variables to $u = r^{-1} - r_{\max}^{-1}$, whence the integral becomes

$$\Delta\theta = 2 \int_0^{\Lambda} \frac{du}{\sqrt{u(\Lambda - u)}} ,$$

where we have introduced the shorthand $\Lambda := r_{\min}^{-1} - r_{\max}^{-1} > 0$. Let us change variables again to $w = \sqrt{u/\Lambda}$, which transforms the integral into an elementary integral

$$\Delta\theta = 4 \int_0^1 \frac{dw}{\sqrt{1 - w^2}} = 2\pi .$$

Hence from Example 4.1 it follows at once that the orbits are periodic.

The minimum allowed energy is $E = V_{\min}$ in which case the orbit is circular with radius $m\ell^2/k$, as found in Example 5.2. Conservation of angular momentum (33) implies that the angular velocity for such an orbit is constant: $\dot\theta = k^2/m^2\ell^3$. It takes a time T for θ to go from 0 to 2π, whence

$$2\pi = \frac{k^2 T}{m^2\ell^3} \implies T = \frac{2\pi m^2 \ell^3}{k^2} . \tag{45}$$

This allows us to estimate the length of the year.

Example 5.4 (Estimating the distance to the Sun)**.** From the knowledge of how long an Earth year is, we can estimate the distance from the Earth to the Sun, assuming the Earth's orbit around the Sun to be circular. For a circular orbit of radius R, we have that

$$\ell^2 = G(m_1 + m_2)R .$$

The corresponding angular velocity is

$$\dot\theta = \frac{\sqrt{G(m_1 + m_2)}}{R^{3/2}} ,$$

whence the period is

$$T = \frac{2\pi R^{3/2}}{\sqrt{G(m_1 + m_2)}} .$$

Solving for R we find

$$R = \left(\frac{G(m_1 + m_2)T^2}{4\pi^2}\right)^{\frac{1}{3}} .$$

For T = 1year, typing

```
((1 year)^2*G*(mass of Earth + mass of Sun)/(4*pi^2))^(1/3) in AU
```

into Google yields 0.999993881 AU, which is just under 1 astronomical unit, as expected.

5.5 The Laplace–Runge–Lenz vector

We have just seen that for $E < 0$ and $\ell \neq 0$, the motion is periodic. We will see in the next lecture that they are indeed ellipses with the centre of mass of the two-body system as a focus. In fact, we will be able to describe the orbits geometrically. We will set the stage by exhibiting yet another conserved quantity in addition to the energy and angular momentum.
Let

$$\boxed{\mathbf{A} := \dot{\boldsymbol{x}} \times \mathbf{L} + V\boldsymbol{x}}$$

denote the **Laplace–Runge–Lenz vector**. We claim that it is conserved. Indeed, using that $\mathbf{L}$ is constant, we find

$$\begin{aligned}\frac{d\mathbf{A}}{dt} &= \ddot{\boldsymbol{x}} \times \mathbf{L} + (\dot{\boldsymbol{x}} \cdot \nabla V)\boldsymbol{x} + V\dot{\boldsymbol{x}} \\ &= m\ddot{\boldsymbol{x}} \times (\boldsymbol{x} \times \dot{\boldsymbol{x}}) + (\dot{\boldsymbol{x}} \cdot \nabla V)\boldsymbol{x} + V\dot{\boldsymbol{x}} \\ &= -\nabla V \times (\boldsymbol{x} \times \dot{\boldsymbol{x}}) + (\dot{\boldsymbol{x}} \cdot \nabla V)\boldsymbol{x} + V\dot{\boldsymbol{x}} \,,\end{aligned}$$

where we have used Newton's equation in the last line. If we now use the explicit expression $V = -k/|\boldsymbol{x}|$, for which $\nabla V = k\boldsymbol{x}/|\boldsymbol{x}|^3$, we obtain

$$\frac{d\mathbf{A}}{dt} = -\frac{k}{|\boldsymbol{x}|^3}\boldsymbol{x} \times (\boldsymbol{x} \times \dot{\boldsymbol{x}}) + \frac{k}{|\boldsymbol{x}|^3}(\dot{\boldsymbol{x}} \cdot \boldsymbol{x})\boldsymbol{x} - \frac{k}{|\boldsymbol{x}|}\boldsymbol{x} \,,$$

which is seen to vanish upon using the identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \,. \tag{46}$$

This identity also allows us to eliminate the cross product from the expression for $\mathbf{A}$:

$$\mathbf{A} = \big(\dot{\boldsymbol{x}} \cdot \boldsymbol{p} + V\big)\boldsymbol{x} - (\dot{\boldsymbol{x}} \cdot \boldsymbol{x})\,\boldsymbol{p} = \big(m|\dot{\boldsymbol{x}}|^2 + V\big)\boldsymbol{x} - m\,(\dot{\boldsymbol{x}} \cdot \boldsymbol{x})\,\dot{\boldsymbol{x}} \,. \tag{47}$$

The existence of this conserved quantity *is* special to the $1/r$ potential and as we will see in the next lecture, it explains that the geometry of the orbits is very simple, namely the orbits are conic sections: ellipses, parabolas and hyperbolas.

The ϵ-symbol (cont'd)

The identity (46) can also be expressed using the ϵ-symbol. Recall that if $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ and $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ is their cross product, then

$$c_i = \epsilon_{ijk} a_j b_k \, .$$

Therefore, if we let $\boldsymbol{d} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$, we have

$$d_i = \epsilon_{ijk} a_j \epsilon_{klm} b_l c_m \, .$$

We now use the following identity

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \, , \tag{48}$$

where we have introduced the Kronecker δ, defined by

$$\delta_{ij} = \begin{cases} 1 \, , & i = j \\ 0 \, , & i \neq j \, . \end{cases}$$

Therefore

$$d_i = a_j c_j b_i - a_j b_j c_i \, ,$$

which is equivalent to (46).

Lecture 6: Kepler's laws of planetary motion

Now it is quite clear to me that there are no solid spheres in the heavens, and those that have been devised by the authors to save the appearances, exist only in the imagination.
— Tycho Brahe

In this lecture we will see that Newton's law of gravitation implies Kepler's laws and discuss in some more detail the bounded orbits in the attractive potential; that is, those orbits with negative energy. We have seen in Example 5.3 that these orbits are closed, and in fact that they are simple closed curves: so that as r goes from $r_{\max}$ to $r_{\min}$ and back to $r_{\max}$, the angle turns by precisely 2π. Our first result is that these orbits are in fact ellipses with the centre of mass as a focus. As we will recall shortly, this is Kepler's first law of planetary motion. In fact, we will prove something stronger: namely that all orbits (whether bounded or not) are given by conic sections; that is, ellipses, parabolas or hyperbolas, depending on the energy.

6.1 Conic sections and planetary orbits

Let us now use the conserved quantities in the Kepler problem to determine the geometry of planetary orbits. The same result will of course hold for the orbits in the Coulomb problem, whether attractive or repulsive.
We recall that since $\boldsymbol{x}\cdot\mathbf{L}=0$, it follows that the motion takes place in the plane perpendicular to $\mathbf{L}$, which, since $\mathbf{L}$ is conserved, is fixed throughout the motion. Since $\mathbf{A}\cdot\mathbf{L}=0$ as well, the Laplace–Runge–Lenz vector lies in the plane of motion. If we let φ denote the angle between $\boldsymbol{x}$ and $\mathbf{A}$, we know that

$$\boldsymbol{x}\cdot\mathbf{A} = |\boldsymbol{x}||\mathbf{A}|\cos\varphi\,,$$

whereas we can compute this explicitly from the definition of $\mathbf{A}$ to obtain

$$\boldsymbol{x}\cdot\mathbf{A} = \boldsymbol{x}\cdot(\dot{\boldsymbol{x}}\times\mathbf{L}) + V|\boldsymbol{x}|^2\,.$$

Using the identity

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) \tag{49}$$

and the explicit form of the potential, we find that

$$\boldsymbol{x}\cdot\mathbf{A} = \mathbf{L}\cdot(\boldsymbol{x}\times\dot{\boldsymbol{x}}) - k|\boldsymbol{x}| = m\ell^2 - k|\boldsymbol{x}|\,.$$

Equating the two expressions for $\boldsymbol{x}\cdot\mathbf{A}$ and rearranging, we find that

$$\boxed{|\boldsymbol{x}|\left(1+\frac{|\mathbf{A}|}{k}\cos\varphi\right) = \frac{m\ell^2}{k}\,,} \tag{50}$$

which is the equation of a conic section with eccentricity $\frac{|\mathbf{A}|}{k}$ and latus rectum $\frac{2m\ell^2}{k}$.

Conic sections

Conic sections (or simply **conics**) in $\mathbb{R}^3$ are planar curves defined by intersecting a right circular double cone with an affine plane. In polar coordinates (r,φ) for the plane, the equation for a conic section is given by

$$r(1+\varepsilon\cos\varphi)=\lambda\,, \tag{51}$$

where $\varepsilon \geq 0$ is called the **eccentricity** and $\lambda > 0$ is called the **semi-latus rectum**. There are three types of (nondegenerate) conics depending on the value of the eccentricity:

$$\begin{cases} \varepsilon < 1\,, & \text{ellipse} \\ \varepsilon = 1\,, & \text{parabola} \\ \varepsilon > 1\,, & \text{hyperbola.} \end{cases}$$

Circles are ellipses with $\varepsilon = 0$.

Earlier we saw that periodic orbits have negative energy, whence we expect that negative energy should be equivalent to the eccentricity < 1. This is indeed the case, as we now show by computing

$$\begin{aligned} |\mathbf{A}|^2 &= |\dot{\boldsymbol{x}}\times\mathbf{L}+\mathrm{V}\boldsymbol{x}|^2 \\ &= m^2|\dot{\boldsymbol{x}}|^2\ell^2 + k^2 + 2m\mathrm{V}\ell^2 \end{aligned}$$

where we have used the identity

$$|\boldsymbol{a}\times\boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2\,, \tag{52}$$

the facts that $\dot{\boldsymbol{x}}$ and $\mathbf{L}$ are perpendicular and that $|\mathbf{L}|^2 = m^2\ell^2$.

Two derivations

Identity (52) can be proved by using (46) and (49):

$$\begin{aligned} |\boldsymbol{a}\times\boldsymbol{b}|^2 &= \boldsymbol{a}\cdot(\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{b})) \\ &= \boldsymbol{a}\cdot\left(|\boldsymbol{b}|^2\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{b}\right) \\ &= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2\,. \end{aligned}$$

Equivalently, it can be proved from equation (48):

$$\begin{aligned} |\boldsymbol{a}\times\boldsymbol{b}|^2 &= \varepsilon_{ijk}a_jb_k\varepsilon_{ilm}a_lb_m \\ &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})a_jb_ka_lb_m \\ &= a_ja_jb_kb_k - a_jb_ja_kb_k \\ &= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2\,. \end{aligned}$$

Continuing with the calculation and rearranging, we find that

$$|\mathbf{A}|^2 - k^2 = 2m\ell^2\left(\tfrac{1}{2}m|\dot{\boldsymbol{x}}|^2 + \mathrm{V}\right) ,$$

which, recognising the energy in the RHS and rearranging again, we rewrite as

$$\varepsilon^2 = 1 + \frac{2m\ell^2}{k^2}\mathrm{E} , \tag{53}$$

where ε is the eccentricity of the orbit. Therefore we see that $\mathrm{E} < 0$ corresponds to ellipses, $\mathrm{E} = 0$ to parabolas and $\mathrm{E} > 0$ to hyperbolas. The circular orbits correspond to zero eccentricity, whence $\mathrm{E} = -k^2/2m\ell^2$ which is the minimum allowed energy, in agreement with the results of Example 5.2.

6.2 Kepler's laws

In perhaps the earliest successful applications of data mining, Kepler studied observations of the planet Mars painstakingly recorded by Brahe over a period of many years. Out of this data Kepler distilled three empirical laws:

I. that the planetary orbits are ellipses with the Sun as a focus,

II. that the planets move in such a way that they sweep equal areas in equal time, and

III. that the period of the orbit is proportional to the $\frac{3}{2}$th power of the length of the semi-major axis of the ellipse (see below).

We have seen above that bounded orbits are indeed ellipses; although in a small correction to Kepler's first law, it is the centre of mass of the Sun/planet system which sits at the focus of the ellipse. For a planet such as the Earth, we saw in Example 3.1 that this correction is almost imperceptible.
The second law follows from the conservation of angular momentum, since this implies that the areal velocity is constant. We also saw in Section 4.3 that this is not a peculiarity of the $1/r$ potential, but in fact a general fact of central force fields.
In proving the third law we will derive below an explicit formula for the period of the orbit, but we can understand why this has to be the case simply as a consequence of the invariance of Newton's equation (for this very particular potential) under a (mechanical) **similarity** transformation.

Example 6.1 (Similarity invariance of Newton's equation)**.** Let $\boldsymbol{x}(t)$ be a solution of Newton's equation:

$$m\ddot{\boldsymbol{x}} = -\frac{k\boldsymbol{x}}{|\boldsymbol{x}|^3} .$$

Then we claim that so is $\boldsymbol{x}_\lambda(t) := \lambda\boldsymbol{x}(\lambda^{-3/2}t)$ for all $\lambda > 0$. Indeed, differentiating once using the chain rule, we obtain

$$\dot{\boldsymbol{x}}_\lambda(t) = \lambda^{-1/2}\dot{\boldsymbol{x}}(\lambda^{-3/2}t) ,$$

where here and in what follows a dot indicates a derivative with respect to the argument. Differentiating again, we obtain

$$\ddot{\boldsymbol{x}}_\lambda(t) = \lambda^{-2}\ddot{\boldsymbol{x}}(\lambda^{-3/2}t) .$$

On the other hand,

$$\frac{-k\boldsymbol{x}_\lambda(t)}{|\boldsymbol{x}_\lambda(t)|^3} = -\lambda^{-2}\frac{k\boldsymbol{x}(\lambda^{-3/2}t)}{|\boldsymbol{x}(\lambda^{-3/2}t)|^2}\,.$$

Comparing we see that $\boldsymbol{x}_\lambda(t)$ indeed solves the equation.

It follows from this result that if $\boldsymbol{x}(t)$ is a planetary orbit with period T and semi-major axis of length a, then $\boldsymbol{x}_\lambda(t)$ is also an orbit but with period $\lambda^{3/2}\mathrm{T}$ and semi-major axis of length λa.

6.3 Elliptical geometry

Since the areal velocity is constant and the orbit is an ellipse, the area swept by the radial vector during one period is the area of the ellipse. Computing the area geometrically we can then determine the period T. More precisely,

$$\text{area of ellipse} = \int_0^{\mathrm{T}} \frac{d\mathrm{A}}{dt}\,dt = \int_0^{\mathrm{T}} \tfrac{1}{2}\ell\,dt = \tfrac{1}{2}\ell\mathrm{T}\,,$$

whence the period is given by

$$\mathrm{T} = \frac{2(\text{area of ellipse})}{\ell}\,. \tag{54}$$

We will now compute the area of the ellipse in terms of physical data. The geometry of the ellipse is depicted in Figure 15, where a and b are the lengths of the **semi-major** and **semi-minor axes**, respectively.

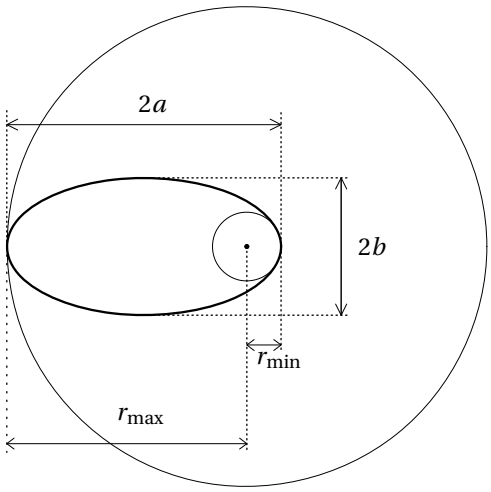


Figure 15: Geometry of the ellipse

It is clear from the picture that $2a = r_{\max} + r_{\min}$. From the polar equation for the ellipse

$$r(1+\varepsilon\cos\varphi) = \lambda\,,$$

we see that $r_{\min}$ occurs when $\varphi = 0$, whence $r_{\min} = \lambda/(1+\varepsilon)$, and $r_{\max}$ occurs when $\varphi = \pi$, whence $r_{\max} = \lambda/(1-\varepsilon)$. This means that the length of the semi-major axis is

given by

$$a = \frac{\lambda}{1-\varepsilon^2}\,, \tag{55}$$

which allows us to write the polar equation for the ellipse as

$$r(1+\varepsilon\cos\varphi) = a(1-\varepsilon^2)\,.$$

The length b of the semi-minor axis of the ellipse is the maximum length of $r\sin\varphi$. Using the polar equation, we see that

$$r\sin\varphi = \frac{a(1-\varepsilon^2)\sin\varphi}{1+\varepsilon\cos\varphi}\,.$$

We find the maximum by differentiating and setting the derivative to zero to obtain

$$\frac{a(1-\varepsilon^2)}{(1+\varepsilon\cos\varphi)^2}\left(\varepsilon+\cos\varphi\right) = 0 \implies \cos\varphi = -\varepsilon\,,$$

which in turn implies that $\sin\varphi = \sqrt{1-\varepsilon^2}$, whence

$$b = a\sqrt{1-\varepsilon^2}\,. \tag{56}$$

We now express both a and b in terms of physical data. From equation (53) we see that

$$1-\varepsilon^2 = \frac{2m\ell^2|\mathrm{E}|}{k^2}\,,$$

and $\lambda = m\ell^2/k$, whence using equation (55) and the above result,

$$a = \frac{k}{2|\mathrm{E}|} \qquad \text{and} \qquad b = \sqrt{\frac{m\ell^2}{2|\mathrm{E}|}}\,.$$

Example 6.2 (Area of ellipse)**.** Let us calculate the area of the ellipse. The equation for an ellipse with semi-major axis of length a and semi-minor axis of length b is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1\,.$$

By symmetry, the area inside the curve is 4 times the area inside the curve and inside the positive quadrant, which can be written as the integral

$$\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx\,.$$

Changing variables to $x = a\sin\theta$, the area is given by

$$\mathrm{A} = 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = \pi ab\,.$$

Finally, we can use equation (54) to solve for the period

$$\mathrm{T} = \frac{2\pi ab}{\ell} = \frac{\pi k}{2|\mathrm{E}|^{3/2}} ,$$

or in terms of the length a of the semi-major axis,

(57)
$$\boxed{\mathrm{T} = 2\pi a^{3/2}\sqrt{\frac{m}{k}} ,}$$

just as Kepler had observed.
Notice finally that this agrees with the case of a circular orbit as in Example 5.2. There we found that the radius of such an orbit was $\mathrm{R} = m\ell^2/k$, whereas equation (33) says that the angular velocity is constant and equal to $\dot\theta = k^2/m^2\ell^3$. The period is $\mathrm{T} = 2\pi/\dot\theta$, which after the dust settles is equal to the expression in (57) with $a = \mathrm{R}$.

Lecture 7: Small oscillations about stable equilibria

In this lecture we begin the study of stable equilibria and show that small displacements around such equilibria result in oscillatory behaviour. We will first see how a one-dimensional particle executes simple harmonic motion in the neighbourhood of a minimum of the potential. We will then generalise this first to three-dimensional motion and then to an arbitrary (but finite) number of degrees of freedom. The further generalisation to an infinite number of degrees of freedom gives rise to the theory of wave motion, which belongs to the second half of this course.

7.1 Equilibria and critical points

Recall from Example 1.1 that an equilibrium point is a point where the force field vanishes. Everyday experience provides a certain intuitive notion of 'stability' of an equilibrium point: if we make a small displacement from equilibrium, the system should remain near the equilibrium point. We will now try to make this intuition precise. We will restrict ourselves for simplicity to the case of conservative force fields. An equilibrium point is then a **critical point** of the potential; that is, a point $\boldsymbol{x}_0$ where $\nabla V(\boldsymbol{x}_0) = \mathbf{0}$.

Let us begin by discussing one-dimensional mechanical systems; that is, a particle moving in a one-dimensional potential. Newton's equation is given by (15). Without loss of generality let us assume that $x = 0$ is a critical point of the potential. Since the potential is defined up to an additive constant, let us choose that constant so that $V(0) = 0$ without loss of generality. Expanding V around 0 we find that

$$V(x) = \tfrac{1}{2}kx^2 + O(x^3)\,, \tag{58}$$

where $k := V''(0)$ is the second derivative of V with respect to x evaluated at $x = 0$. For small deviations from equilibrium we may truncate this expansion to second order and assume that $V(x) = \frac{1}{2}kx^2$. Newton's equation is then

$$m\ddot{x} = -kx\,. \tag{59}$$

There are three cases depending on whether k is positive, zero or negative. If $k = 0$ we say that the critical point is **degenerate** and we cannot say anything about its stability. Otherwise we have a **non-degenerate** critical point: a local minimum if $k > 0$ and a local maximum if $k < 0$. In the former case the force tends to restore the system to equilibrium and we say that the equilibrium point is **stable**. In the latter case the force tends to push away from equilibrium and we say that the equilibrium point is **unstable**. We see that Hooke's Law (see Example 2.4) is not particular to springs, but in fact generic for small displacements around a stable equilibrium point. As we saw in Section 2.2, this gives rise to simple harmonic motion.

We saw in Section 4.3 that conservation of angular momentum allows us to reduce the dynamics of a particle moving in a conservative central force field to an effective one-dimensional problem. The radial equation takes the form of Newton's equation with an effective potential $V_{\text{eff}}(r)$. A critical point of the effective potential is a radius r_0 where $V'_{\text{eff}}(r_0) = 0$. This means that circular orbits with radius r_0 are possible. Such circular orbits will be stable if r_0 is a minimum of the effective potential; that is, if $V''_{\text{eff}}(r_0) > 0$. Let us assume that this is the case. If we let $r = r_0 + \rho$, where ρ, assumed small, is the displacement from the circular orbit, the equation for small

radial oscillations can be obtained from (35) by expanding the effective potential about r_0 up to second order

$$V_{\text{eff}}(r) = V_{\text{eff}}(r_0) + \tfrac{1}{2}V''_{\text{eff}}(r_0)\rho^2 + \cdots$$

Since $m\ddot{r} = m\ddot{\rho}$, equation (35) becomes

$$m\ddot{\rho} = -V''_{\text{eff}}(r_0)\rho \,,$$

which describes simple harmonic motion with frequency $\sqrt{V''_{\text{eff}}(r_0)/m}$.

Example 7.1 (Radial oscillations in the Kepler problem)**.** We saw in Example 5.2 that the Kepler problem, for which the effective potential is given by equation (43), admits circular orbits for $r_0 = m\ell^2/k$. Let us calculate the period of radial oscillations around the circular orbit. We start by calculating $V''_{\text{eff}}(r_0)$. Differentiating V_{eff} twice we find

$$V''_{\text{eff}}(r) = -\frac{2k}{r^3} + \frac{3m\ell^2}{r^4} \,,$$

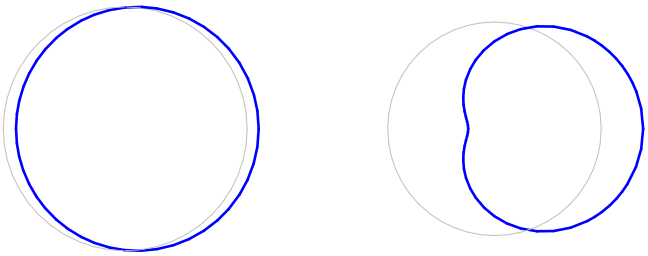
whence evaluating at r_0 and simplifying, we obtain

$$V''_{\text{eff}}(r_0) = \frac{k^4}{m^3\ell^6} > 0 \,.$$

Hence r_0 is indeed a minimum and the circular orbit is stable under small displacements. The period of radial oscillations is given by

$$T = 2\pi\sqrt{\frac{m}{V''_{\text{eff}}(r_0)}} = \frac{2\pi m^2\ell^3}{k^2} \,.$$

Curiously, this is precisely the period (45) of the circular orbit! This means that perturbing the circular orbit we still get a closed orbit, illustrated in Figure 16. (This is also the case with the space oscillator potential.)



(a) small perturbation (b) large perturbation

Figure 16: Perturbations of a circular orbit in the Kepler potential. (The thinner line is the unperturbed circular orbit.)

7.2 Three-dimensional potential motion

Now let us consider a particle of mass m moving in three dimensions subject to a general potential function $V : \mathbb{R}^3 \to \mathbb{R}$. Let $\boldsymbol{x}_0$ be an equilibrium point; that is, a critical point of the potential. Recall that this means that $\nabla V(\boldsymbol{x}_0) = \mathbf{0}$. Critical points come in several types, depending on the "shape" of the potential function near the critical point. This can be quantified by expanding the potential function in a Taylor series around the critical point.

Taylor expansions

Let $V : \mathbb{R}^3 \to \mathbb{R}$ be an infinitely differentiable function. The first few terms in the Taylor expansion of V about the point $\boldsymbol{x}_0 \in \mathbb{R}^3$ take the form

$$V(\boldsymbol{x}) = V(\boldsymbol{x}_0) + (\boldsymbol{x}-\boldsymbol{x}_0)\cdot\nabla V(\boldsymbol{x}_0) + \tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)\cdot H(\boldsymbol{x}-\boldsymbol{x}_0) + \cdots \tag{60}$$

where the **hessian matrix** H has entries

$$H_{ij} = \left.\frac{\partial^2 V}{\partial x^i \partial x^j}\right|_{x_0} . \tag{61}$$

Since V is differentiable, H is symmetric: $H_{ij} = H_{ji}$.

We will make two simplifying assumptions without any loss of generality. First, we will assume that the critical point $\boldsymbol{x}_0$ is the origin. This can be achieved simply by changing coordinates: $\boldsymbol{x} \mapsto \boldsymbol{x} - \boldsymbol{x}_0$. Second, we will assume that the potential function vanishes at the critical point. This can be achieved by subtracting $V(\boldsymbol{x}_0)$ from the potential, which is allowed since it is only the gradient of V that enters Newton's equation. With these two assumptions, the Taylor (or now actually MacLaurin) series (60) of V around $\mathbf{0}$ is given to second order by

$$V(\boldsymbol{x}) = \tfrac{1}{2}\boldsymbol{x}\cdot H\boldsymbol{x} + \cdots , \tag{62}$$

where H is the hessian matrix of V at $\mathbf{0}$.
For small displacements from equilibrium, $|\boldsymbol{x}|$ is assumed to be small, whence we can approximate the potential by the lowest term of its Taylor series, giving rise to a linear force field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\mathbf{F} = -\nabla V = -H\boldsymbol{x} , \tag{63}$$

and Newton's equation becomes

$$m\ddot{\boldsymbol{x}} = -H\boldsymbol{x} \qquad \text{or equivalently} \qquad \ddot{\boldsymbol{x}} = -\tfrac{1}{m}H\boldsymbol{x} . \tag{64}$$

Let us assume for a moment that H is **diagonal**:

$$H = \begin{pmatrix} k_1 & & \\ & k_2 & \\ & & k_3 \end{pmatrix} .$$

In this case Newton's equation decouples into three independent equations

$$m\ddot{x}_i = -k_i x_i , \quad \text{for } i = 1,2,3.$$

If $k_i > 0$, then x_i executes simple harmonic function with frequency $\omega_i = \sqrt{k_i/m}$. Let us assume that $k_i > 0$ for $i = 1,2,3$. Then we can solve each decoupled equation as done above, to obtain

$$x_i(t) = \mathrm{A}_i \sin\left(\omega_i t + \varphi_i\right) .$$

Whereas each x_i is a periodic function, with period $2\pi/\omega_i$, the overall motion need not be periodic. Indeed, motion is periodic if for some $\mathrm{T} > 0$, $\boldsymbol{x}(t+\mathrm{T}) = \boldsymbol{x}(t)$ for all t. In particular, $\boldsymbol{x}(\mathrm{T}) = \boldsymbol{x}(0)$. This means that for each i, $x_i(\mathrm{T}) = x_i(0)$. This is the case if and only if T divides each of the periods $2\pi/\omega_i$ evenly; that is, $\mathrm{T} = 2\pi n_i/\omega_i$ for some positive integers n_i. In other words, this is true if and only if $\omega_i/\omega_j = n_i/n_j$; that is, if the frequencies are rationally related.

7.3 Normal modes and characteristic frequencies

In general, H will not be diagonal, but nevertheless, it is a fundamental result in Linear Algebra that any symmetric matrix, e.g., H, is **diagonalisable**.

The spectral theorem for symmetric matrices

Let H be a symmetric $\mathrm{N} \times \mathrm{N}$ matrix. We say that a *nonzero* vector $\boldsymbol{v} \in \mathbb{R}^{\mathrm{N}}$ is a **(real) eigenvector** of H with **(real) eigenvalue** $\lambda \in \mathbb{R}$, if $\mathrm{H}\boldsymbol{v} = \lambda \boldsymbol{v}$. It is a basic fact from Linear Algebra that H has N linearly independent eigenvectors $\boldsymbol{v}_i$, $i = 1,\ldots,\mathrm{N}$ with eigenvalues λ_i, not necessarily distinct. Furthermore we can choose the $\boldsymbol{v}_i$ to be pairwise orthogonal $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$ for $i \neq j$ and normalised to $|\boldsymbol{v}_i| = 1$. This means that the matrix S whose i-th column is $\boldsymbol{v}_i$ is **orthogonal**: $\mathrm{S}^t\mathrm{S} = \mathrm{I}$, where I is the $\mathrm{N} \times \mathrm{N}$ identity matrix and ^{t} denotes the transpose operation. If we let D be the diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{N}}$, then by construction, $\mathrm{HS} = \mathrm{SD}$.

Applying this to our three-dimensional example, there exists an orthogonal 3×3 matrix S such that

$$\mathrm{H} = \mathrm{SDS}^t \qquad \text{where} \qquad \mathrm{D} = \begin{pmatrix} k_1 & & \\ & k_2 & \\ & & k_3 \end{pmatrix} . \tag{65}$$

The linearised Newton's equation (64) becomes

$$m\ddot{\boldsymbol{x}} = -\mathrm{SDS}^t \boldsymbol{x} .$$

Acting on both sides of this equation from the left with S^t and using that S is orthogonal, we find

$$m\mathrm{S}^t\ddot{\boldsymbol{x}} = -\mathrm{DS}^t \boldsymbol{x} .$$

In terms of the new variable $\boldsymbol{y} = \mathrm{S}^t\boldsymbol{x}$, the equation of motion becomes

$$m\ddot{\boldsymbol{y}} = -\mathrm{D}\boldsymbol{y} ,$$

where we have used that S is independent of time. This reduces the problem to the case of H diagonal, but in terms of $\boldsymbol{y}$ instead of $\boldsymbol{x}$. Indeed, if $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, the equation

decouples into three one-dimensional equations for each y_i:

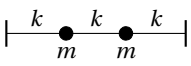
$$\ddot{y}_i = \frac{k_i}{m} y_i \ .$$

Assuming that $\mathbf{0}$ is a minimum of the potential, so that the hessian matrix is positive-definite, $k_i > 0$ for all i. Then the above equations have oscillatory solutions

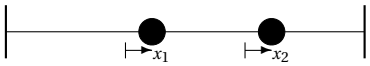
$$y_i(t) = \mathrm{A}_i \sin\left(\omega_i t + \varphi_i\right) \ ,$$

where $\omega_i^2 = k_i/m$. Once we have determined $\boldsymbol{y}(t)$ we recover $\boldsymbol{x}(t)$ by changing variables back: $\boldsymbol{x}(t) = \mathrm{S}\boldsymbol{y}(t)$. The variables $y_i(t)$ are known as the **normal modes** of the system and the frequencies ω_i are known as the **characteristic frequencies**. In summary, small oscillations about a stable critical point are linear combinations of independent one-dimensional oscillators. This does not preclude the system to exhibit seemingly complicated behaviour as we will see below.

7.4 Coupled one-dimensional oscillators



Consider an idealised one-dimensional mechanical system consisting of two point masses each of mass m connected by springs to each other and to two fixed ends. We will neglect gravity, friction and the mass of the springs. The springs obey Hooke's law with spring constant k. We assume that the system is at equilibrium when the springs are relaxed, and we want to study the system around equilibrium; that is, we wish to study small displacements of the masses. We let x_i for $i = 1,2$ denote the displacements from equilibrium for each of the two point masses, as shown below.



Then the potential energy due to the springs is the sum of the potential energies of each of the springs:

$$\begin{aligned} \mathrm{V} &= \tfrac{1}{2}k\,x_1^2 + \tfrac{1}{2}k\,(x_2 - x_1)^2 + \tfrac{1}{2}k\,x_2^2 \\ &= k\left(x_1^2 + x_2^2 - x_1 x_2\right) \ . \end{aligned}$$

The kinetic energy is given by

$$\mathrm{T} = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 \ .$$

The equations of motion are then, for $i = 1,2$,

$$\frac{d}{dt}\frac{\partial \mathrm{T}}{\partial \dot{x}_i} = -\frac{\partial \mathrm{V}}{\partial x_i} \ .$$

Explicitly, we have the following coupled system of second order ordinary differential equations:

$$\begin{aligned} m\ddot{x}_1 &= -2kx_1 + kx_2 \\ m\ddot{x}_2 &= -2kx_2 + kx_1 \ . \end{aligned}$$

Let us write this in matrix form. We introduce a column vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then the above system of equations becomes

$$\ddot{\boldsymbol{x}} = -\omega^2 \mathrm{K}\,\boldsymbol{x}\,, \tag{66}$$

where K is the matrix

$$\mathrm{K} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and where we have introduced the notation

$$\omega := \sqrt{\frac{k}{m}}\,.$$

Notice that K is symmetric, hence it can be diagonalised by an orthogonal transformation. Let us find its eigenvalues and its eigenvectors. The characteristic polynomial of K is given by

$$\chi_{\mathrm{K}}(\lambda) = \begin{Vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{Vmatrix} = (2-\lambda)^2 - 1 = (\lambda-1)(\lambda-3)\,,$$

from which it follows that it has as roots $\lambda = 1, 3$. The normalised eigenvectors corresponding to these eigenvalues are

$$\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

respectively. We build the following matrix S out of the normalised eigenvectors

$$\mathrm{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

One can check that S is orthogonal: $\mathrm{S}^t = \mathrm{S}^{-1}$. One can also check that

$$\mathrm{K} = \mathrm{S}\,\mathrm{D}\,\mathrm{S}^t\,,$$

where D is the diagonal matrix of eigenvalues

$$\mathrm{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Inserting this expression into equation (66), we see that

$$\ddot{\boldsymbol{x}} = -\omega^2 \mathrm{S}\mathrm{D}\mathrm{S}^t \boldsymbol{x}\,.$$

In terms of the new variables

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathrm{S}^t \boldsymbol{x}\,,$$

the equation of motion (66) becomes

$$\ddot{\boldsymbol{y}} = -\omega^2 \mathrm{D}\,\boldsymbol{y}\,. \tag{67}$$

Because the matrix D is diagonal, the equations of motion for the new variables y_i are now decoupled:

$$\ddot{y}_1 = -\omega^2 y_1 \qquad \text{and} \qquad \ddot{y}_2 = -3\omega^2 y_2\,.$$

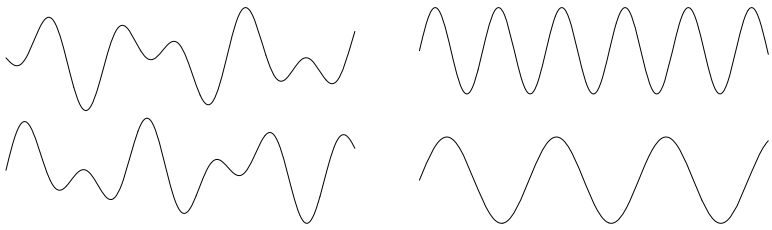
One can now easily solve these equations,

$$y_1(t) = A_1 \sin(\omega_1 t + \varphi_1)$$
$$y_2(t) = A_2 \sin(\omega_2 t + \varphi_2) ,$$

where $\omega_1 = \omega$, $\omega_2 = \sqrt{3}\omega$ and A_i and φ_i are constants to be determined from the initial conditions. The physical variables in the original problem are the displacements x_i of each of the point masses. They can be found in terms of the new decoupled variables y_i simply by inverting the change of variables (67). Explicitly,

$$x_1(t) = \frac{A_1}{\sqrt{2}} \sin(\omega_1 t + \varphi_1) + \frac{A_2}{\sqrt{2}} \sin(\omega_2 t + \varphi_2)$$
$$x_2(t) = \frac{A_1}{\sqrt{2}} \sin(\omega_1 t + \varphi_1) - \frac{A_2}{\sqrt{2}} \sin(\omega_2 t + \varphi_2) .$$

The variables y_i are the normal modes of the system and they decouple the equations of motion. Their virtue is that they reduce an interacting (i.e., coupled) mechanical system around equilibrium to a set of independent free oscillators. Each of these free oscillators are mathematical constructs: the normal modes do not generally correspond to the motion of any of the masses in the original system, but they nevertheless possess a certain "physicality" and it is fruitful to work with them as if they were physical. The original physical variables can then be understood as linear combinations of the normal modes as we saw above. The frequencies ω_i of the normal modes are the characteristic frequencies of the mechanical system. In particle physics, for example, the elementary particles are the normal modes and their masses are the characteristic frequencies.
To illustrate the simplification in the dynamics which results from considering the normal modes, in Figure 17 we have sketched the motion of the two masses in the problem and of the two normal modes, with time running horizontally to the right. Notice also that although the motion of each of the normal modes is periodic, the

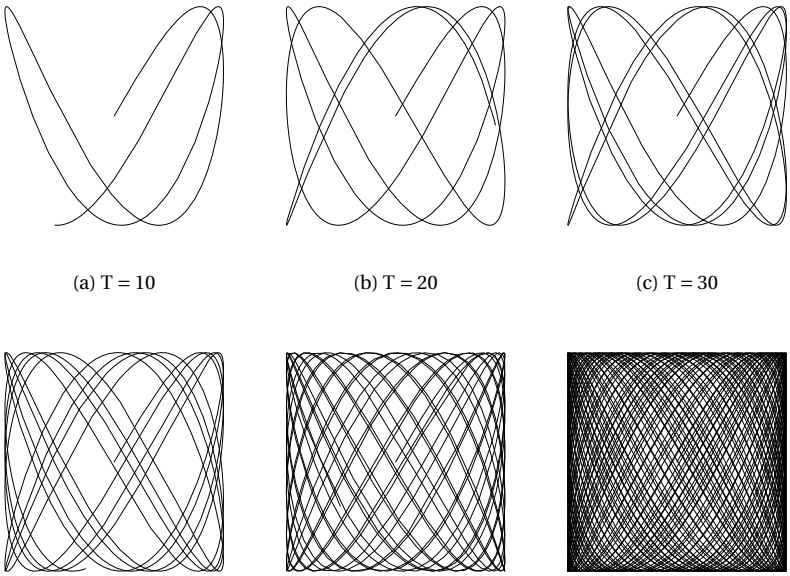


(a) Point masses (b) Normal modes

Figure 17: Dynamics of point masses and normal modes.

system as a whole is not. This is due to the fact that the characteristic frequencies are not rational multiples of each other. Indeed, if we were to plot the trajectory of the system in the plane, with the trajectory of one of the point masses along the x-axis and the trajectory of the other point mass along the y-axis, we see that the orbit never repeats, and that we end up filling up the available configuration space. In Figure 18 we have plotted the cumulative trajectory of the system after letting it run

for T units of time, for different values of T. As you can see, as T grows the system has visited more and more points in the available configuration space. Asymptotically, as $T \to \infty$, the system will have visited the whole available space.



(a) T = 10 (b) T = 20 (c) T = 30

(d) T = 50 (e) T = 100 (f) T = 300

Figure 18: Trajectory of the mechanical system at different times.

7.5 Near-equilibrium dynamics

In this section we will consider a more general mechanical system near equilibrium. Consider a mechanical system whose configuration space is N-dimensional euclidean space $\mathbb{R}^N$. For example, it could be a system of n point particles in d dimensions, and then $N = nd$. In the previous section we discussed the case of a one-dimensional system consisting of two point particles, so that the configuration space was $\mathbb{R}^2$.
The potential energy is given by a function $V : \mathbb{R}^N \to \mathbb{R}$. The configurations of mechanical equilibrium are those for which the gradient of the potential vanishes. Hence let us consider one such equilibrium configuration $\boldsymbol{q}_0 \in \mathbb{R}^N$:

$$\nabla V|_{\boldsymbol{q}_0} = \mathbf{0} \,.$$

Because the potential energy is only defined up to an additive constant, we are free to choose it such that $V(\boldsymbol{q}_0) = 0$. We can therefore expand the potential function V about $\boldsymbol{q}_0$ and the first contribution will be quadratic:

$$\begin{aligned} V(\boldsymbol{q}) &= V(\boldsymbol{q}_0) + \nabla V|_{\boldsymbol{q}_0} \cdot (\boldsymbol{q} - \boldsymbol{q}_0) + \tfrac{1}{2}(\boldsymbol{q} - \boldsymbol{q}_0) \cdot H(\boldsymbol{q} - \boldsymbol{q}_0) + \cdots \\ &= \tfrac{1}{2}(\boldsymbol{q} - \boldsymbol{q}_0) \cdot H(\boldsymbol{q} - \boldsymbol{q}_0) \,, \end{aligned}$$

where $H : \mathbb{R}^N \to \mathbb{R}^N$ is the hessian of V at $\boldsymbol{q}_0$, thought of as a symmetric linear transformation of $\mathbb{R}^N$. Explicitly, in terms of the canonical basis $\{\boldsymbol{e}_i\}$ for $\mathbb{R}^N$, then let $\boldsymbol{q} = \sum_i q_i \boldsymbol{e}_i$ define some coordinates q_i for the configuration space. Then relative to this basis the hessian of V has matrix elements

$$H_{ij} = \boldsymbol{e}_i \cdot H(\boldsymbol{e}_j) = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}_0} ,$$

which shows manifestly that it is symmetric: $H_{ij} = H_{ji}$. Let us define $\boldsymbol{x} = \boldsymbol{q} - \boldsymbol{q}_0$ to be the displacements about equilibrium. These will be our dynamical variables. The potential energy in the quadratic approximation is given by

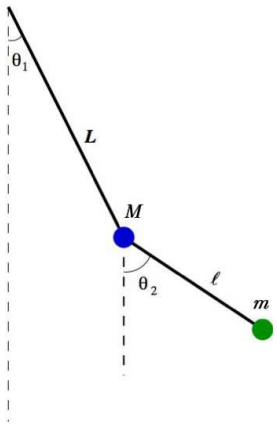
$$V = \tfrac{1}{2}\boldsymbol{x} \cdot H(\boldsymbol{x}) = \tfrac{1}{2}\sum_{i,j} H_{ij} x_i x_j .$$

We will make the assumption that the kinetic energy is quadratic in the velocities $\dot{\boldsymbol{x}}$:

$$T = \tfrac{1}{2}\dot{\boldsymbol{x}} \cdot M(\dot{\boldsymbol{x}}) = \tfrac{1}{2}\sum_{i,j} M_{ij} \dot{x}_i \dot{x}_j ,$$

where the **mass matrix** M is assumed to be symmetric and positive-definite; that is, all its eigenvalues are positive.

Example 7.2 (The double pendulum)**.** Consider the double pendulum shown in the figure.



It consists of a pendulum of length ℓ with a bob of mass m hanging from a pendulum of length L which has a bob of mass M. We assume that the motion is constrained to a vertical plane. Let θ_1 and θ_2 denote the angular displacements from equilibrium of each of the bobs as shown in the figure below. We will assume that they are small.
The kinetic energy of the first bob is $T_1 = \frac{1}{2}ML^2\dot{\theta}_1^2$, whereas for the second bob, one has to take into account that θ_2 is measured relative to the first bob, which is moving. Hence we must add the velocities, so that the kinetic energy of the second bob is $T_2 = \frac{1}{2}m(L\dot{\theta}_1 + \ell\dot{\theta}_2)^2$. The total kinetic energy is therefore

$$T = T_1 + T_2 = \tfrac{1}{2}ML^2\dot{\theta}_1^2 + \tfrac{1}{2}m(L\dot{\theta}_1 + \ell\dot{\theta}_2)^2 .$$

This is of the torm $T = \frac{1}{2}\sum_{i,j} M_{ij}\dot{\theta}_i\dot{\theta}_j$, where the mass matrix is given by

$$[M_{ij}] = \begin{pmatrix} (M+m)L^2 & m\ell L \\ m\ell L & m\ell^2 . \end{pmatrix}$$

It is clearly positive-definite, since the kinetic energy can be written as a sum of squares.

We will now analyse the dynamics of small displacements from equilibrium following the following prescription:

1. we will standardise the kinetic energy by diagonalising and normalising the mass matrix; and

2. we will then diagonalise the potential energy and solve for the normal modes and characteristic frequencies of this system.

Both steps make use of the spectral theorem for symmetric transformations. Because M is symmetric, there is an orthogonal matrix S_1 such that $M' = S_1^t M S_1$ is diagonal with positive entries. Let D_1 be the diagonal matrix whose entries are the (positive) square roots of the diagonal entries of M'. In other words, $M' = D_1^2$. We can therefore write

$$M = S_1 D_1^2 S_1^t = (S_1 D_1)\,(S_1 D_1)^t \ ,$$

where we have used that $D_1^t = D_1$ since it is diagonal. Introduce then the following variables

$$\boldsymbol{z} = (S_1 D_1)^t \boldsymbol{x} = D_1 S_1^t \boldsymbol{x} \ .$$

We can invert this change of variables as follows:

$$\boldsymbol{x} = S_1 D_1^{-1} \boldsymbol{z} \ ,$$

where we have used that S_1 is orthogonal, so that $S_1^t = S_1^{-1}$. This change of variables accomplishes the first step outlined above, since in terms of $\boldsymbol{z}$, the kinetic energy becomes simply

$$T = \tfrac{1}{2}|\dot{\boldsymbol{z}}|^2 = \tfrac{1}{2}\sum_i (\dot{z}_i)^2 \ .$$

Similarly, the potential energy has become

$$V = \tfrac{1}{2}\boldsymbol{z}\cdot K\boldsymbol{z} \ ,$$

where the matrix K is defined by

$$K = D_1^{-1} S_1^t H S_1 D_1^{-1} \ ,$$

which is clearly symmetric since H and D_1 are. Therefore we can find a second orthogonal matrix S_2 such that $D := S_2^t K S_2$ is diagonal. Let us define a new set of variables

$$\boldsymbol{y} = S_2^t \boldsymbol{z} \ ,$$

relative to which the kinetic energy remains simple

$$T = \tfrac{1}{2}|S_2\,\dot{\boldsymbol{y}}|^2 = \tfrac{1}{2}|\dot{\boldsymbol{y}}|^2 \ ,$$

since orthogonal matrices preserve norms, and the potential energy diagonalises

$$V = \tfrac{1}{2}\boldsymbol{y}\cdot D\,\boldsymbol{y} \ .$$

Because D is diagonal, the equations of motion of the $\boldsymbol{y}$ are decoupled:

$$\ddot{\boldsymbol{y}} = -D\,\boldsymbol{y} \ ,$$

whence the $\boldsymbol{y}$ are the normal modes of the system. Let D have entries

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} \ ,$$

Then the equations of motion for the normal modes are

$$\ddot{y}_i = -\lambda_i\, y_i \ .$$

We can distinguish three types of solutions:

1. ($\lambda_i > 0$) The solution is oscillatory with characteristic frequency $\omega_i = \sqrt{\lambda_i}$:

$$y_i(t) = A_i \sin(\omega_i\, t + \varphi_i)\ .$$

2. ($\lambda_i = 0$) The solution is linear

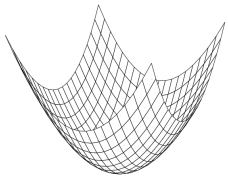
$$y_i(t) = a_i + b_i\, t\ .$$

   Such a normal mode is said to be a **zero mode**, since it has zero characteristic frequency.

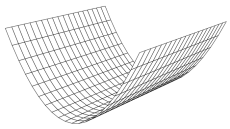
3. ($\lambda_i < 0$) The solution is exponential

$$y_i(t) = A_i \exp\left(\sqrt{|\lambda_i|}\, t\right) + B_i \exp\left(-\sqrt{|\lambda_i|}\, t\right)\ .$$

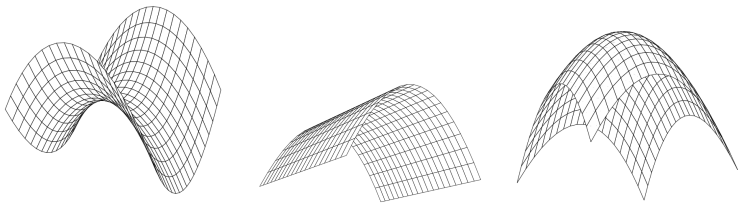
If all eigenvalues λ_i are positive the equilibrium point is said to be **stable**, if they are all non-negative then it is **semistable**, whereas if there is a negative eigenvalue, then the equilibrium is **unstable**. The signs of the eigenvalues of the matrix D agree with the sign of the eigenvalues of the hessian matrix of the potential at the equilibrium point. The different types of equilibria are illustrated in Figure 19, which shows the behaviour of the potential function around an equilibrium point in the simple case of a two-dimensional configuration space. The existence of zero modes is symptomatic of *flat directions* in the potential along which the system can evolve without spending any energy. This usually signals the existence of some continuous symmetry in the system. In the Figure we see that the semistable equilibrium point indeed has a flat direction along which the potential is constant. In other words, translation along the flat direction is a symmetry of the potential function.



(a) stable



(b) semistable



(c) unstable

Figure 19: Different types of equilibrium points.